Define $u(t) = \exp(t \log x + (1-t)\log y)$.
$u(0) = y$, $u(1) = x$, $|u(t)| = |x|^t |y|^{1-t} \leq 1$
for all $t \in [0,1]$. Hence, $|u'(t)| \leq |\log x - \log y|$.

$$|x-y| = \left|\int_0^1 u'(t)\,dt\right| \leq \sup_{t\in[0,1]} |u'(t)| \leq |\log x - \log y|$$