KAIST POW 2014-18

2012[redacted] Jimin Park

October 7, 2014

Lemma 1. *When n is odd, $\forall A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, if $A + A^T$ is invertible, then* $\text{nullity}(A) \leq (n-1)/2$.

Proof. Assume that $\text{nullity}(A) \geq (n+1)/2$. Since $\text{nullity}(A) = \text{nullity}(A^T)$, $\text{nullity}(A) + \text{nullity}(A^T) = 2 \times \text{nullity}(A) = n + 1 > n$, so $\exists x \in \mathbb{R}^n$ such that $x \neq 0$ and $x \in \ker(A) \cap \ker(A^T)$. Then $(A + A^T)x = Ax + A^T x = 0$, which shows that $A + A^T$ is not invertible.

Therefore, if $A + A^T$ is invertible, then $\text{nullity}(A) \leq (n-1)/2$. □

Problem. *Let A and B be $n \times n$ real matrices for an odd integer n. Prove that if both $A + A^T$ and $B + B^T$ are invertible, then $AB \neq 0$.*

Proof. By the Lemma 1, $\text{nullity}(A) \leq (n-1)/2$ and $\text{nullity}(B) \leq (n-1)/2$. By the rank-nullity theorem, $\dim \text{im}(B) \geq (n+1)/2 > (n-1)/2 = \dim \ker(A)$, so $\exists x \in \mathbb{R}^n$, $Bx \notin \ker(A)$ and $ABx \neq 0$. Therefore $AB \neq 0$. □