Zeros of a polynomial

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$$p(x) = x^n + x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

be a polynomial. Prove that if $p(z) = 0$ *for a complex number* z*, then*

$$|z| \leq 1 + \sqrt{\max\left(|a_0|, |a_1|, |a_2|, \cdots, |a_{n-2}|\right)}.$$

Solution. Let $a \equiv \max\left(|a_0|, \cdots, |a_{n-2}|\right)$. Since $-z^n - z^{n-1} = a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$,

$$\begin{aligned} |z|^{n-1} |z+1| &= \left|a_{n-2}z^{n-2} + \cdots + a_1 z + a_0\right| \\ &\leq a(|z|^{n-2} + \cdots + |z| + 1) = a\frac{|z|^{n-1} - 1}{|z| - 1}. \end{aligned}$$

If $|z| \leq 1$, we are done; thus, we may assume that $|z| > 1$. Then, the above inequality becomes

$$(|z| - 1)^2 \leq |z+1|\left(|z| - 1\right) \leq a\frac{|z|^{n-1} - 1}{|z|^{n-1}} \leq a,$$

and therefore, $|z| \leq 1 + \sqrt{a}$ follows. $\square$