2014-19 Two Complex Numbers

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Problem. Prove that for two non-zero complex numbers x and y, if $|x|, |y| \leq 1$, then $|x - y| \leq |\log x - \log y|$.

Proof. We will prove it by showing $|x - y|^2 \leq |\log x - \log y|^2$. Let $\theta \in [0, \pi]$ the smallest positive angular difference between x and y. By the cosine rule

$$|x - y|^2 = (|x| - |y|)^2 + 2|x||y|\,(1 - \cos\theta) \leq (|x| - |y|)^2 + 2 - 2\cos\theta,$$

and by the properties of logarithm

$$|\log x - \log y|^2 \geq (\log|x| - \log|y|)^2 + \theta^2 \geq (|x| - |y|)^2 + \theta^2.$$

With the Taylor series we know $2 - 2\cos\theta \leq \theta^2$, which completes the proof.