

POW 2014-12 Rational ratios in a triangle

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Problem) Determine all triangles ABC such that all of $\frac{AB}{BC}$, $\frac{BC}{CA}$, $\frac{CA}{AB}$, $\frac{\angle A}{\angle B}$, $\frac{\angle B}{\angle C}$, $\frac{\angle C}{\angle A}$ are rational.

Answer) $\triangle ABC$ must be an equilateral triangle

proof) Step I. $\cos \angle A$, $\cos \angle B$, $\cos \angle C$ are rational.

By applying the 2nd law of cosines, $\cos \angle A = \frac{CA^2 + AB^2 - BC^2}{2 \cdot CA \cdot AB} = \frac{1}{2} \left\{ \frac{CA}{AB} + \frac{AB}{CA} - \frac{BC}{CA} \cdot \frac{BC}{AB} \right\}$

Since $\frac{AB}{BC}$, $\frac{BC}{CA}$, $\frac{CA}{AB}$ are rationals, we can know that $\cos \angle A$ is also rational.

Similarly, we can know $\cos \angle B$ and $\cos \angle C$ are also rational.

Step II. $\exists d_1, d_2, d_3 \in \mathbb{Q}$ s.t. $\angle A = d_1 \pi$, $\angle B = d_2 \pi$, $\angle C = d_3 \pi$ ($0 < d_1, d_2, d_3 < 1$, $d_1 + d_2 + d_3 = 1$)

It can be easily known from the condition, $\frac{AB}{BC}$, $\frac{BC}{CA}$, $\frac{CA}{AB}$ are rational.

Step III. When $d \in \mathbb{Q}$ and $\cos(d\pi) \in \mathbb{Q}$, $\cos(d\pi) \in \left\{ -1, -\frac{1}{2}, 0, \frac{1}{2}, 1 \right\}$

Suppose $2 \cos(d\pi) = \frac{a}{b}$ ($\gcd(a, b) = 1$)

From $2 \cos(2d\pi) = [2 \cos(d\pi)]^2 - 2 = \frac{a^2 - 2b^2}{b^2}$ ($\gcd(a^2 - 2b^2, b^2) = 1$),

we can know that if $b \neq \pm 1$, then in $2 \cos(d\pi)$, $2 \cos(2d\pi)$, $2 \cos(2^2 d\pi)$, ...

the denominators get bigger and bigger.

However, for $\forall k \in \mathbb{Z}_0^+$, $2 \cos(2^k d\pi) \in \left\{ 2 \cos\left(\frac{1}{\pi}\pi\right), 2 \cos\left(\frac{2}{\pi}\pi\right), \dots, 2 \cos\left(\frac{2^n}{\pi}\pi\right) \right\}$ when $d = \frac{m}{\pi}$

It means that the sequence $(2 \cos(2^k d\pi))_{k \in \mathbb{Z}_0^+}$ may admit at most n different values.

Thus, it will run into a cycle.

This contradicts the observation above that its denominators necessarily tend to infinity.

Consequently, the only way out is $b = \pm 1$. Hence, $\cos(d\pi) \in \left\{ 1, -\frac{1}{2}, 0, \frac{1}{2}, 1 \right\}$

Step IV. Find the triangle satisfying the conditions

From step II, $0 < d_1 \pi < \pi$. Now, apply step III, possible d_1 are $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$.

Similarly, $d_1, d_2, d_3 \in \left\{ \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \right\}$.

Since $d_1 + d_2 + d_3 = 1$, the only solution is $d_1 = d_2 = d_3 = \frac{1}{3}$ (i.e. $d_1 \pi = d_2 \pi = d_3 \pi = \frac{\pi}{3}$)

Hence, $\triangle ABC$ must be an equilateral triangle. \square