KAIST POW 2014-13

서울대학교 어수강

[[PROBLEM]]

Prove that, for any unit vectors $v_1, v_2, \cdots, v_n$ in $\mathbb{R}^n$, there exists a unit vector w in $\mathbb{R}^n$ such that $\langle w, v_i \rangle \le n^{-\frac{1}{2}}$ for all $i = 1, 2, \cdots, n$. (Here, $\langle \bullet, \bullet \rangle$ is a usual scalar product in $\mathbb{R}^n$.)

[[PROOF]]

[LEMMA]

For any vector v, w in $\mathbb{R}^n$,

if $\| v+w \| < \sqrt{\| v \|^2 + \| w \|^2}$ then $\| v-w \| > \sqrt{\| v \|^2 + \| w \|^2}$.

[PROOF OF LEMMA]

Let v, w in $\mathbb{R}^n$ be given.

Suppose that $\| v+w \| < \sqrt{\| v \|^2 + \| w \|^2}$.

Since $\| v+w \|^2 = \| v \|^2 + \| w \|^2 + 2\langle v, w \rangle < \| v \|^2 + \| w \|^2$, $\langle v, w \rangle < 0$.

Thus $\| v-w \|^2 = \| v \|^2 + \| w \|^2 + 2\langle v, -w \rangle = \| v \|^2 + \| w \|^2 - 2\langle v, w \rangle > \| v \|^2 + \| w \|^2$.

Hence $\| v-w \| > \sqrt{\| v \|^2 + \| w \|^2}$.

Now, we go on to the main proof.

[MAIN PROOF]

Denote $V = \{v_1, v_2, \cdots, v_n\}$, $V_i = V - \{v_i\}$ and $W = \langle V \rangle$, $W_i = \langle V_i \rangle$ for all $1 \le i \le n$.

[STEP1] Suppose $\dim W \le n-1$. Then $\dim W^\perp \ge 1$, since $\dim \mathbb{R}^n = \dim W + \dim W^\perp$.

Thus there exist unit vector $w \in W^\perp$. Since $v_1, v_2, \cdots, v_n \in W$ and $w \in W^\perp$, $\langle w, v_i \rangle = 0$ for all $1 \le i \le n$. We are done.

[STEP2] Suppose that $\dim W = n$. Then $\dim W_i = n-1$ for all $1 \le i \le n$.

Thus $\dim W_i^\perp = 1$ for all $1 \le i \le n$. Hence there exist unit vector $w_i \in W_i^\perp$ for all $1 \le i \le n$.

[STEP3] By our lemma, $\| w_1 + w_2 \| \ge \sqrt{2}$ or $\| w_1 - w_2 \| \ge \sqrt{2}$.

(Since, if $\| w_1 + w_2 \| < \sqrt{2}$ then $\| w_1 - w_2 \| > \sqrt{2}$.)

WLOG, We may assume that $\| w_1 + w_2 \| \ge \sqrt{2}$. (Since, if $w_2 \in W_2^\perp$ then $-w_2 \in W_2^\perp$. So we can choose $w_2 \in W_2^{\perp}$ whose satisfies $\| w_1 + w_2 \| \geq \sqrt{2}$ in STEP2.)

Similarly, by our lemma, $\| (w_1 + w_2) + w_3 \| \geq \sqrt{3}$ or $\| (w_1 + w_2) - w_3 \| \geq \sqrt{3}$.
WLOG, We may assume that $\| w_1 + w_2 + w_3 \| \geq \sqrt{3}$.
Continuing in this way, WLOG, we have $\left\| \sum_{j=1}^{n} w_j \right\| \geq \sqrt{n}$.

[STEP4] Let $w = \dfrac{\sum_{i=1}^{n} w_i}{\left\| \sum_{i=1}^{n} w_i \right\|}$. Then w is a unit vector.

Moreover, $\langle w, v_i \rangle = \left\langle \dfrac{\sum_{j=1}^{n} w_j}{\left\| \sum_{j=1}^{n} w_j \right\|}, v_i \right\rangle$

$= \dfrac{1}{\left\| \sum_{j=1}^{n} w_j \right\|} \sum_{j=1}^{n} \langle w_j, v_i \rangle$

$= \dfrac{1}{\left\| \sum_{j=1}^{n} w_j \right\|} \langle w_i, v_i \rangle$ (Since $w_j \in W_j^{\perp}$ and $v_i \in W_i$, $\langle w_j, v_i \rangle = 0$ for all $j \neq i$.)

$\leq \dfrac{1}{\left\| \sum_{j=1}^{n} w_j \right\|}$ (Since w_i, v_i are unit vectors, $\langle w_i, v_i \rangle \leq 1$.)

$\leq n^{-\frac{1}{2}}$ (Since $\left\| \sum_{j=1}^{n} w_j \right\| \geq \sqrt{n}$) for all $1 \leq i \leq n$.

Which is what we wanted. ■■