KAIST POW 2014-10 : Inequality with pi

2014학번 장기정

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1 Problem

Prove that, for any sequences of real number $\{a_n\}$ and $\{b_n\}$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \le \pi \left(\sum_{m=1}^{\infty} a_m^2\right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2}$$

2 Solution

Lemma 1. *For each $m \in \mathbb{N}$,*

$$\sum_{n=1}^{\infty}\frac{\sqrt{m}}{\sqrt{n}(m+n)} < \pi.$$

Proof. Use geometric approach : Denote points $O = (0,0), A_n(\sqrt{m}, \sqrt{n})$ and S be the area of the circle centered at $(0,0)$ with radius $\sqrt{m}$ in first quadrant. Let B_n be intersection between the circle $x^2 + y^2 = m$ and OA_n. Then let S_n be area of part of circle $OB_{n-1}B_n$ and let C_n be the intersection of OA_{n-1} and vertical line passing B_n. Then

$$\frac{\pi m}{4} = S = \sum_{n=1}^{\infty} S_n > \sum_{n=1}^{\infty} |\Delta OB_nC_n|$$

$$= \sum_{n=1}^{\infty}\left(\frac{|OB_n|}{|OA_n|}\right)^2 |OA_{n-1}A_n|$$

$$= \sum_{n=1}^{\infty}\frac{m}{m+n}\cdot\frac{1}{2}\sqrt{m}(\sqrt{n}-\sqrt{n-1})$$

$$= \sum_{n=1}^{\infty} \frac{m\sqrt{m}(\sqrt{n} - \sqrt{n-1})}{2(m+n)}$$

$$> \sum_{n=1}^{\infty} \frac{m\sqrt{m}}{4\sqrt{n}(m+n)}.$$

Then we earn the statement by multiplying $4/m$ to each side of inequality. $\square$

Write

$$\frac{a_m b_n}{m+n} = \frac{\sqrt[4]{m}}{\sqrt[4]{n}\sqrt{m+n}} a_m \cdot \frac{\sqrt[4]{n}}{\sqrt[4]{m}\sqrt{m+n}} b_n,$$

then by Cauchy-Schwarz Inequality,

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \sqrt{\sum_{m,n=1}^{\infty} \frac{\sqrt{m}}{\sqrt{n}(m+n)} a_m^2} \cdot \sqrt{\sum_{m,n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{m}(m+n)} b_n^2}$$

$$= \sqrt{\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{\sqrt{m}}{\sqrt{n}(m+n)} \right) a_m^2} \cdot \sqrt{\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{\sqrt{n}}{\sqrt{m}(m+n)} \right) b_n^2}$$

$$\leq \pi \sqrt{\sum_{m=1}^{\infty} a_m^2} \sqrt{\sum_{n=1}^{\infty} b_n^2}.$$

Last inequality holds by Lemma.