POW 2014 - 09 Product of Series

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Problem) For integer $n \geq 1$, define

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}, \quad b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}$$

Prove that $a_n b_n$ is an integer.

Proof) Let, $a_0 = \frac{1}{0!} + \sum_{k=1}^{\infty} \frac{k^0}{k!}$, $b_0 = \frac{1}{0!} + \sum_{k=1}^{\infty} (-1)^k \frac{k^0}{k!}$

Step I. Find a_0, a_1, b_0, b_1 (Use Taylor series of e^x : $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$)

$$a_0 = \frac{1}{0!} + \sum_{k=1}^{\infty} \frac{1}{k!} = e^1, \quad a_1 = \sum_{k=0}^{\infty} \frac{k}{k!} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} = e^1$$

$$b_0 = \frac{1}{0!} + \sum_{k=1}^{\infty} (-1)^k \frac{1}{k!} = e^{-1}, \quad b_1 = \sum_{k=0}^{\infty} (-1)^k \frac{k}{k!} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{(k-1)!} = -e^{-1}$$

Step II. Find a_n, b_n ($n \geq 1$, integer)

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!} = \sum_{k=1}^{\infty} \frac{k^n}{k!} = \sum_{k=1}^{\infty} \frac{k^{n-1}}{(k-1)!} = \sum_{k=1}^{\infty} \frac{\{(k-1)+1\}^{n-1}}{(k-1)!} = \binom{n-1}{0} \sum_{k=1}^{\infty} \frac{(k-1)^{n-1}}{(k-1)!} + \binom{n-1}{1} \sum_{k=1}^{\infty} \frac{(k-1)^{n-2}}{(k-1)!} + \cdots + \binom{n-1}{n-2} \sum_{k=1}^{\infty} \frac{(k-1)^1}{(k-1)!} + \binom{n-1}{n-1} \sum_{k=1}^{\infty} \frac{1}{(k-1)!}$$

$$= \binom{n-1}{0} a_{n-1} + \binom{n-1}{1} a_{n-2} + \cdots + \binom{n-1}{n-2} a_1 + \binom{n-1}{n-1} a_0 \quad \cdots\cdots (*)$$

$$b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!} = \sum_{k=1}^{\infty} (-1)^k \frac{k^n}{k!} = \sum_{k=1}^{\infty} (-1)^k \frac{k^{n-1}}{(k-1)!} = \sum_{k=1}^{\infty} (-1)^k \frac{\{(k-1)+1\}^{n-1}}{(k-1)!}$$

$$= \binom{n-1}{0} \sum_{k=1}^{\infty} (-1)^k \frac{(k-1)^{n-1}}{(k-1)!} + \binom{n-1}{1} \sum_{k=1}^{\infty} (-1)^k \frac{(k-1)^{n-2}}{(k-1)!} + \cdots + \binom{n-1}{n-2} \sum_{k=1}^{\infty} (-1)^k \frac{(k-1)^1}{(k-1)!} + \binom{n-1}{n-1} \sum_{k=1}^{\infty} (-1)^k \frac{1}{(k-1)!}$$

$$= -\binom{n-1}{0} b_{n-1} - \binom{n-1}{1} b_{n-2} - \cdots - \binom{n-1}{n-2} b_1 - \binom{n-1}{n-1} b_0 \quad \cdots\cdots (*')$$

Step III. a_n, b_n (integer time of e^1, e^{-1} each for integer $n \geq 0$)

(1) Since $a_0 = a_1 = e^1$, $b_0 = e^{-1}$, $b_1 = -e^{-1}$, the statement is verified for $n = 0, 1$

(2) Assume this statement satisfies for $0 \leq n \leq k-1$. Then, we can know that the statement satisfies also for $n = k$ since the coefficients in $(*)$, $(*')$ of step II are all integers.

By mathematical induction, a_n, b_n are integer time of e^1, e^{-1} each for integer $n \geq 0$.

($\therefore$) Consequently, for any integer $n \geq 1$, $\exists\, s, t \in \mathbb{Z}$ s.t. $a_n = se^1$ and $b_n = te^{-1}$ by step III.

Hence, $a_n b_n = se^1 \cdot te^{-1} = st \in \mathbb{Z}$. ($a_n b_n$ is an integer.)