KAIST POW 2014-07 : Subsequence

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1 Problem

Let $a_1, a_2, \cdots$ be an infinite sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that for every positive constant c, there exists an infinite sequence $i_1 < i_2 < i_3 < \cdots$ of positive integers such that $|i_n - cn^3| = O(n^2)$ and $\sum_{n=1}^{\infty} \left(a_{i_n}(a_1^{1/3} + ... + a_{i_n}^{1/3}) \right)$ converges.

2 Solution

Let $\sum_{n=1}^{\infty} a_n = A$. Since c is positive, there exists a positive integer N that satisfies both $c(3N^2 - 3N + 1) \geq 2$ and $c(N-1)^3 > N - 1$. Then, for any positive integer $n \geq N$, there exists at least one integer in the (real) interval $(c(n-1)^3, cn^3]$. Define $i_n (n \geq N)$ to be the least (to avoid ambiguity) integer $c(n-1)^3 < m \leq cn^3$ that satisfies $a_m = \min\{a_i | c(n-1)^3 < i \leq cn^3, i \in \mathbb{N}\}$. And define remaining components $i_1, ... i_{N-1}$ by

$$i_k = \max\{k, \lfloor ck^3 \rfloor\} (1 \leq k < N).$$

We'll see that such defined sequence $\{i_n\}$ satisfies the conditions in the problem.

(i) $\{i_n\}$ is increasing : Enough to show that $i_n < i_{n+1}$ for every $n \in \mathbb{N}$.

if $n < N$ and $\lfloor cn^3 \rfloor = \lfloor c(n+1)^3 \rfloor$, $c(3n^3 + 3n + 1) < 1$ so $cn^3 = \sum_{k=0}^{n-1} c(3k^2 + 3k + 1) < n$. hence $a_n = \max\{\lfloor cn^3 \rfloor, n\} = n < a_{n+1}$. if $\lfloor cn^3 \rfloor < \lfloor c(n+1)^3 \rfloor$, $a_n = \max\{\lfloor cn^3 \rfloor, n\} < \max\{\lfloor c(n+1)^3 \rfloor, n+1\} = a_{n+1}$.

If $n = N$, either $i_{N-1} = N - 1 < c(N-1)^3 < i_N$ or $i_{N-1} = \lfloor c(N-1)^3 \rfloor < i_N$.

if $n \geq N$, $i_n \leq cn^3 < i_{n+1}$ for $n \geq N$. Hence, we can conclude that the sequence is increasing.
(ii) $|i_n - cn^3| = O(n^2)$: if $n \geq N$, $|i_n - cn^3| = cn^3 - i_n < c(n^3 - (n-1)^3) = c(3n^2 - 3n + 1) < 3cn^2$ hence $|i_n - cn^3| = O(n^2)$. if $n < N$, if $cn^3 < n$, $|i_n - cn^3| < n <$ and if $cn^3 \geq n$, $|i_n - cn^3| = cn^3 - \lfloor cn^3 \rfloor < 1$ so $|i_n - cn^3| = O(n^2)$.
(iii) $\sum_{n=1}^{\infty} \left(a_{i_n}(a_1^{1/3} + ... + a_{i_n}^{1/3}) \right)$ converges : For $n \geq N$, define a sequence $S_n = \sum_{c(n-1)^3 < k \leq cn^3} a_k$. Note that $\sum_{n=N}^{\infty} S_n = \sum_{k > c(N-1)^3} a_k$ converges. Then by definition,

$$a_{i_n} \leq \frac{\sum_{c(n-1)^3 < k \leq cn^3} a_k}{\sum_{c(n-1)^3 < k \leq cn^3} 1} = \frac{S_n}{\lfloor cn^3 \rfloor - \lfloor c(n-1)^3 \rfloor}$$

$$\leq \frac{S_n}{c(3n^2 - 3n + 1) - 1} \leq \frac{2S_n}{c(3n^2 - 3n + 1)} \leq \frac{2S_n}{cn^2}.$$

(last two inequalities holds since $c(3N^2 - 3N + 1) \geq 2$ and $2n^2 - 3n + 1 \geq 0$ for $n \geq 1$.) Meanwhile, by Cauchy-Schwarz inequality,

$$a_1^{1/3} + ... + a_{i_n}^{1/3} \leq \left(i_n^2 \sum_{k=1}^{i_n} a_k \right)^{1/3} \leq \left((cn^3)^2 \sum_{k=1}^{i_n} a_k \right)^{1/3} \leq c^{2/3} A^{1/3} n^2.$$

Combining two inequalities,

$$a_{i_n}(a_1^{1/3} + ... + a_{i_n}^{1/3}) \leq \frac{2S_n}{cn^2} c^{2/3} A^{1/3} n^2 = (2A^{1/3}/c^{1/3}) S_n$$

while

$$\sum_{n=N}^{\infty} (2A^{1/3}/c^{1/3}) S_n = (2A^{1/3}/c^{1/3}) \sum_{n=N}^{\infty} S_n$$

converges. Therefore, $\sum_{n=1}^{\infty} \left(a_{i_n}(a_1^{1/3} + ... + a_{i_n}^{1/3}) \right)$ converges by comparison test.
By (i), (ii) and (iii), we can complete the proof.