KAIST POW 2014-11 : Subsets of a countably infinite set

2014학번 장기정

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1 Problem

Prove or disprove that every uncountable collection of subsets of a countably infinite set must have two members whose intersection has at least 2014 elements.

2 Solution

The statement is true.
Let X be a countably infinite set, $\mathcal{A}$ be an uncountable collection of subsets of X. Assume that every two members of $\mathcal{A}$ has intersection having less than 2014 elements. Define a collection $\mathcal{B} = \{A \subset X||A| \leq 2013\}$. Note that $\mathcal{B}$ is a countable collection of subsets of X, hence $\mathcal{C} = \mathcal{A} - \mathcal{B}$ is an uncountable collection of subsets of X, that every two members of $\mathcal{C}$ has intersection having less than 2014 elements.
Choose any 2014-element subset A of X. If there exist two elements B, C of $\mathcal{C}$, $|B \cap C| \geq 2014$ hence contradicts with the assumption. So, there exists up to unique element of $\mathcal{C}$ that includes A : hence we can define a function from $\{A \subset X||A| = 2014, \exists B \in \mathcal{C}[A \subset B]\}$ to $\mathcal{C}$ by mapping subset to element that containing such subset. Moreover, this mapping is surjective, since for every $B \in \mathcal{C}$, there exists subset D of B that $|D| = 2014$, from $|B| \geq 2014$ from definition. hence $\mathcal{C}$, which is image of surjective function in $\{A \subset X||A| = 2014, \exists B \in \mathcal{C}[A \subset B]\} \subset \{A \subset X||A| = 2014\}$ which is countable set, is countable set. But it contradicts with above discussion that $\mathcal{C}$ is

uncountable, hence we can conclude that the initial assumption is false, so the statement is true.