POW 2014-05

Let n, k be positive integers and let $A_1, A_2, \ldots, A_n$ be $k \times k$ real matrices. Prove or disprove that $\det(\sum_{i=1}^{n} A_i^T A_i) \geq 0$.

sol)

Let $A = \sum_{i=1}^{n} A_i^T A_i$ and $p(t) = \det(A - tI)$

Assume that $p(0) < 0$.
Also, $\lim_{t \to -\infty} p(t) = \infty$.
By intermediate value theorem, $\exists \lambda < 0$ such that $p(\lambda) = 0$.
Then, there exists an eigenvector X such that $AX = \lambda X$ and $X \neq O$.

$$\lambda(X^T X) = X^T(\lambda X) = X^T(AX) = X^T((\sum_{i=1}^{n} A_i^T A_i)X) = \sum_{i=1}^{n}(X^T A_i^T A_i X) = \sum_{i=1}^{n}(A_i X)^T(A_i X)$$

$$\Rightarrow \lambda = \frac{\sum_{i=1}^{n}(A_i X)^T(A_i X)}{X^T X}$$

Since $X^T X > 0$ and $(A_i X)^T(A_i X) \geq 0$, $\lambda \geq 0$.
It contradicts to the hypothesis.
Thus, $p(0) \geq 0 \Leftrightarrow \det A \geq 0$