2014-02 Series 20130716 황성호

Problem) Determine all $l \in N$ s.t. $\sum_{n=1}^{\infty} \frac{n^3}{(n+1)(n+2)\cdots(n+l)}$ converges and if it converges, then compute its value

Sol) If $l \le 4$, $\frac{n^3}{(n+1)\cdots(n+l)} \ge \frac{n^3}{5!\, n^l} = \frac{1}{5!}\frac{1}{n^{l-3}}$ $(\because (n+1)\cdots(n+l) \le 2n\, 3n \cdots (l+1)n \le 5!\, n^l)$

By p-series test, since $l-3 \le 1$, $\sum_{n=1}^{\infty} \frac{n^3}{5!\, n^l}$ diverges.

By Comparison test, $\sum_{n=1}^{\infty} \frac{n^3}{(n+1)\cdots(n+l)}$ also diverges.

If $l \ge 5$, $\frac{n^3}{(n+1)(n+2)\cdots(n+l)} \le \frac{n^3}{n^l} = \frac{1}{n^{l-3}}$

By p-series test, since $l-3>1$, $\sum_{n=1}^{\infty} \frac{n^3}{n^l}$ converges.

By Comparison test, $\sum_{n=1}^{\infty} \frac{n^3}{(n+1)(n+2)\cdots(n+l)}$ also converges.

Ans: $l \ge 5$ $(l \in N)$

Sol) Step I. $\sum_{n=1}^{\infty} \frac{1}{(n+1)\cdots(n+l)} = \sum_{n=1}^{\infty} \frac{1}{(l-1)}\left\{\frac{1}{2\cdots l} - \frac{1}{3\cdots(l+1)}\right\} = \frac{1}{l-1}\cdot\frac{1}{2\cdots l} = \frac{1}{(l-1)\, l!}$

Step II. $\sum_{n=1}^{\infty} \frac{(n+1)}{(n+1)\cdots(n+l)} = \sum_{n=1}^{\infty} \frac{1}{(n+2)\cdots(n+l)} = \frac{1}{(l-2)}\frac{1}{3\cdots l} = \frac{2}{(l-2)\, l!}$

Step III. $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{(n+1)\cdots(n+l)} = \sum_{n=1}^{\infty} \frac{1}{(n+3)\cdots(n+l)} = \frac{1}{(l-3)}\frac{1}{4\cdots l} = \frac{6}{(l-3)\, l!}$

Step IV. $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)(n+3)}{(n+1)\cdots(n+l)} = \sum_{n=1}^{\infty} \frac{1}{(n+4)\cdots(n+l)} = \frac{1}{(l-4)}\frac{1}{5\cdots l} = \frac{24}{(l-4)\, l!}$

$$\sum_{n=1}^{\infty} \frac{n^3}{(n+1)\cdots(n+l)} = \sum_{n=1}^{\infty} \frac{(n+1)(n+2)(n+3) - 6(n+1)(n+2) + 7(n+1) - 1}{(n+1)\cdots(n+l)}$$
$$= \frac{24}{(l-4)\, l!} - \frac{36}{(l-3)\, l!} + \frac{14}{(l-2)\, l!} - \frac{1}{(l-1)\, l!}$$
$$= \frac{l^2(l+5)}{(l-4)(l-3)(l-2)(l-1)\, l!}$$

$(\therefore)$ $$\sum_{n=1}^{\infty} \frac{n^3}{(n+1)\cdots(n+l)} = \frac{l(l+5)}{(l-4)(l-3)(l-2)(l-1)(l-1)!} \quad (l \ge 5)$$