POW 2014-01 Uniform convergence

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Consider $X_{n,x}$ for each positive integer n and $x \in I = [0,1]$, a random variable having a binomial distribution $b(n,x)$. Then for any $\delta > 0$, by Chebyshev's inequality,

$$P\left(\left|\frac{X_{n,x}}{n} - x\right| > \delta\right) \leq \frac{Var(X_{n,x}/n)}{\delta^2} = \frac{x(1-x)}{n\delta^2} \leq \frac{1}{4n\delta^2}.$$

Note that this inequality holds regardless of x.

Since its domain is compact, f is uniformly countinuous. That is, for any $\epsilon > 0$, there is $\delta > 0$ such that $|f(y) - f(z)| < \epsilon$ for any $y, z \in I$ with $|y - z| < \delta$. Therefore,

$$p_{n,x} := P\left(\left|f\left(\frac{X_{n,x}}{n}\right) - f(x)\right| > \epsilon\right) \leq P\left(\left|\frac{X_{n,x}}{n} - x\right| > \delta\right) \leq \frac{1}{4n\delta^2}.$$

Moreover, since $M = \|f\|_\infty < \infty$, the expectation is bounded irrespective of x:

$$E\left[\left|f\left(\frac{X_{n,x}}{n}\right) - f(x)\right|\right] \leq \epsilon \cdot (1 - p_{n,x}) + (2M) \cdot p_{n,x} \leq \epsilon + \frac{M}{2n\delta^2}.$$

Note that $E\left[f\left(\frac{X_{n,x}}{n}\right)\right] = B_n(f;x)$ by construction.

Finally, using $|E[X] - E[Y]| \leq E[|X - Y|]$ and regardlessness of x, we can find $N \in \mathbb{Z}^+$ for any $\epsilon > 0$ such that for all $n \geq N$,

$$\|B_n(f) - f\|_\infty \leq \epsilon + \frac{M}{2n\delta^2} < 2\epsilon.$$

This exactly means $B_n(f)$ converges to f uniformly. $\square$