# KAIST POW 2013-22

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**Problem.** *The only field automorphism of* $\mathbb{R}$ *is* $f(x) = x$.

*Proof.* Let $f$ be a field automorphism of $\mathbb{R}$, so it satisfies $f(a+b) = f(a) + f(b)$, $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{R}$.

(1) $\forall a \in \mathbb{R}$, $f(a) = f(0+a) = f(0) + f(a)$, so $f(0) = 0$.

(2) Since $f$ is a bijective map, $\forall x \in \mathbb{R}$, if $x \neq 0$, $f(x) \neq f(0) = 0$.

(3) $f(1) = f(1 * 1) = f(1)^2$, and by (2), $f(1) = 1$.

(4) If $f(i) = i$ $(i \in \mathbb{N})$, then $f(i+1) = f(i) + 1 = i + 1$, so $f(i+1) = i+1$, and by the mathematical induction, $f(i) = i$ for all $i \in \mathbb{N}$.

(5) $\forall a \in \mathbb{R}$, $0 = f(0) = f(a+(-a)) = f(a)+f(-a)$, so $\forall a \in \mathbb{R}$, $f(-a) = -f(a)$.

(6) Similarly, $\forall a \in \mathbb{R}$, if $a \neq 0$, $f(a^{-1}a) = f(1) = 1$, so $\forall a \in \mathbb{R}$, $a \neq 0 \Rightarrow f(a^{-1}) = f(a)^{-1}$.

(7) By (4) and (5), if $i \in \mathbb{Z}$, $(i < 0)$ $f(i) = -f(-i) = -(-i) = i$, $(i = 0)$ $f(i) = 0 = i$, or $(i > 0)$ $f(i) = i \quad \because i \in \mathbb{N}$, so $\forall i \in \mathbb{Z}, f(i) = i$.

(8) By (6) and (7), if $i \in \mathbb{Z}$, $i \neq 0 \Rightarrow f(1/i) = 1/i$.

(9) If $q \in \mathbb{Q}$, then $\exists a, b \in \mathbb{Z}$ such that $b \neq 0$, so by (7) and (8), $f(q) = f(a/b) = a/b = q$, so $\forall q \in \mathbb{Q}$, $f(q) = q$.

(10) $\forall a \in \mathbb{R}$, $a > 0 \Rightarrow \exists r \in \mathbb{R}$, $r \neq 0 \wedge r^2 = a$, so by (2), $f(a) = f(r^2) = f(r)^2 > 0$. And by (5), $\forall a \in \mathbb{R}$, $a < 0 \Rightarrow f(a) = -f(-a) < 0$. Therefore, $\forall a \in \mathbb{R}$, $a > 0 \Leftrightarrow f(a) > 0$ and $a < 0 \Leftrightarrow f(a) < 0$.

(11) By (10) and (5), $\forall a, b \in \mathbb{R}$, $a < b \Leftrightarrow b-a > 0 \Leftrightarrow f(b-a) > 0 \Leftrightarrow f(b)-f(a) > 0 \Leftrightarrow f(a) < f(b)$.

(12) By (9) and (11), $\forall a \in \mathbb{R}, \forall q \in \mathbb{Q}$, $a > q \Leftrightarrow f(a) > q$, and similarly, $\forall a \in \mathbb{R}, \forall q \in \mathbb{Q}$, $a < q \Leftrightarrow f(a) < q$.

(13) $\forall a, b \in \mathbb{R}$, if $a < b$, then $\exists n \in \mathbb{N}$, $n(b-a) > 2$ (ex: $n = \lceil \frac{2}{b-a} \rceil$) so that $\exists m \in \mathbb{N}$, $na < m < nb$ (ex: $m = \lfloor na + 2 \rfloor$), and $a < \frac{m}{n} < b$. Therefore, $\forall a, b \in \mathbb{R}$, $\exists q \in \mathbb{Q}$, $a < q < b$.

(14) By (9) and (13) $\forall x \in \mathbb{R}, x < f(x) \Rightarrow \exists q \in \mathbb{Q}, x < q < f(x)$, which contradicts with (12). Similarly, $f(x) < x \Rightarrow \exists q \in \mathbb{Q}$, $f(x) < q < x$, which also contradicts with (12). Therefore, $\forall x \in \mathbb{R}$, there can be only one case, $x = f(x)$, so $\forall x \in \mathbb{R}$, $x = f(x)$.

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