Unique inverse

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POW2013-21. *Let* $f(z) = z + e^{-z}$. *Prove that, for any real number* $\lambda > 1$, *there exists a unique* $w \in H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ *such that* $f(w) = \lambda$.

Solution. Let $z = a + bi$ with $a, b \in \mathbb{R}$ and $a > 0$. Then, $\operatorname{Im} f(z) = b - e^{-a} \sin b = 0 \Leftrightarrow b = 0$ because $e^a > 1$. Now, restrict f to reals. $f(0) = 1$ and $f'(x) = 1 - e^{-x} > 0$ implies that f is strictly increasing, and $f((0, \infty)) = (1, \infty)$. Consequently, for given $\lambda > 1$, there exists the unique $x \in \mathbb{R}^+$ with $f(x) = \lambda$, while $f(z) \notin \mathbb{R}$ if $\operatorname{Re}(z) > 0$ and $z \notin \mathbb{R}$. $\square$