POW 2013-20 Eigenvalues of Hermitian matrices

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November 18, 2013

Problem. Let $A, B, C = A+B$ be $N \times N$ Hermitian matrices. Let $\alpha_1 \geq \cdots \geq \alpha_N$, $\beta_1 \geq \cdots \geq \beta_N$, $\gamma_1 \geq \cdots \geq \gamma_N$ be the eigenvalues of A, B, C, respectively. For any $1 \leq i, j \leq N$ with $i + j - 1 \leq N$, prove that

$$\gamma_{i+j-1} \leq \alpha_i + \beta_j$$

Solution.

Lemma 1. Let M be an $N \times N$ Hermitian matrix. Let $\lambda_1 \geq \cdots \geq \lambda_N$ be the eigenvalues of M. Then for each $1 \leq k \leq N$

(1) There is a subspace V of $\mathbb{C}^N$ with $\dim V = k$ such that for any $v \in V$ with $|v| = 1$, $\lambda_k \leq v^* M v$.
(2) There is a subspace W of $\mathbb{C}^N$ with $\dim W = N - k + 1$ such that for any $w \in W$ with $|w| = 1$, $w^* M w \leq \lambda_k$.

Proof. By Spectral Theorem, there is an orthonormal basis $u_1, \cdots, u_N$ of $\mathbb{C}^N$ such that for each $1 \leq i \leq N$, u_i is an eigenvector corresponding to λ_i. For each $v = \sum_{i=1}^N a_i u_i \in \mathbb{C}^N$,

$$v^* M v = v^* \left(\sum_{i=1}^N a_i \lambda_i u_i \right) = \sum_{i=1}^N \lambda_i |a_i|^2.$$

Let V be the subspace spanned by $u_1, \cdots, u_k$. Then for any $v = \sum_{i=1}^k a_i u_i \in V$ with $|v| = 1$, $v^* M v = \sum_{i=1}^k \lambda_i |a_i|^2 \geq \lambda_k$. This proves (1).

Let W be the subspace spanned by $u_k, \cdots, u_N$. Note that the dimension of W is $N - k + 1$. Then for any $w = \sum_{i=k}^N b_i u_i \in W$ with $|w| = 1$, $w^* M w = \sum_{i=k}^N \lambda_i |b_i|^2 \leq \lambda_k$. This proves (2). □

By the Lemma, there are subspaces U, V, W of $\mathbb{C}^N$ such that

1. $\dim U = N - i + 1$, and for any $v \in U$ with $|v| = 1$, $v^* A v \leq \alpha_i$.
2. $\dim V = N - j + 1$, and for any $v \in V$ with $|v| = 1$, $v^* B v \leq \beta_j$.
3. $\dim W = i + j - 1$, and for any $v \in W$ with $|v| = 1$, $\gamma_{i+j-1} \leq v^* C v$.

Since $\dim(U) + \dim(V) - \dim(U \cap V) = \dim(U + V) \leq N$, $N - (i + j - 2) \leq \dim(U \cap V)$. Because $N < \dim(U \cap V) + \dim(W)$, the space $U \cap V \cap W$ is nontrivial. So there is $v \neq 0$ such that $v^* A v \leq \alpha_i$, $v^* B v \leq \beta_j$, and $\gamma_{i+j-1} \leq v^* C v$. This implies

$$\gamma_{i+j-1} \leq v^*(A + B)v = v^* A v + v^* B v \leq \alpha_i + \beta_j.$$

□