POW 2013-17 Repeated numbers

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Problem. A real sequence $x_1, x_2, x_3, \cdots$ satisfies the relation $x_{n+2} = x_{n+1} + x_n$ for $n = 1, 2, 3, \cdots$. If a number r satisfies $x_i = x_j = r$ for some i and j $(i \neq j)$, we say that r is a repeated number in this sequence. Prove that there can be more than 2013 repeated numbers in such a sequence, but it is impossible to have infinitely many repeated numbers.

Solution. Let $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \cdots$ be the sequence of Fibonacci numbers. That is, $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. It is obvious that the sequence strictly increases after the first term(F_1). In particular, the sequence $F_0, F_1, F_2, \cdots$ has only one repeated number. Also, we know that $F_{n+1}/F_n \to (1+\sqrt{5})/2$ as $n \to \infty$.

Consider the following sequence:

$$F_{4027}, -F_{4026}, F_{4025}, -F_{4024}, \cdots, F_3, -F_2, F_1, F_0, F_1, \cdots, F_{4026}, F_{4027}, \cdots$$

In this sequence, each subsequent number is the sum of the previous two. Also, $F_1, F_3, \cdots, F_{4025}, F_{4027}$ are 2014 distinct repeated numbers.

Now let us show that there is no real sequence $x_1, x_2, x_3, \cdots$ satisfying $x_{n+2} = x_{n+1} + x_n$ with infinitely many repeated numbers.

Let $x_1, x_2, x_3, \cdots$ be a real sequence satisfying $x_{n+2} = x_{n+1} + x_n$. If $x_2 = 0$, then the sequence $x_2, x_3, x_4, \cdots$ is actually a constant multiple of the Fibonacci sequence, hence having only finitely many repeated numbers.

Let us assume that x_2 is nonzero. Also without loss of generality, we may assume that x_2 is positive. If $x_2 < 0$, then just consider the sequence obtained by negating each terms. By induction, it can be easily shown that $x_n = F_{n-2}x_1 + F_{n-1}x_2$ for each $n \in \mathbb{N}$(For convenience, let $F_{-1} = 1$).

If $x_1/x_2 > -(1+\sqrt{5})/2$, there is $N \in \mathbb{N}$ such that $x_1/x_2 > -F_{n-1}/F_{n-2}$ for all $n \geq N$. That is, $x_n = F_{n-2}x_1 + F_{n-1}x_2 > 0$ for all $n \geq N$. This implies $x_{N+1} < x_{N+2} < x_{N+3} < \cdots$. So $x_1, \cdots, x_N$ are the only possible repeated numbers.

If $x_m (m > N)$ is a repeated number, then $x_m = x_k$ for some $k \leq N$. Therefore there are only finitely many repeated numbers.

Similarly, if $x_1/x_2 < -(1 + \sqrt{5})/2$, there is $N \in \mathbb{N}$ such that $x_n = F_{n-2}x_1 + F_{n-1}x_2 < 0$ for all $n \geq N$. This implies $x_{N+1} > x_{N+2} > x_{N+3} > \cdots$, and so there are only finitely many repeated numbers.

Finally let us show that if $x_1/x_2 = -(1 + \sqrt{5})/2$, there is no repeated number in the sequence at all. Suppose that $x_1/x_2 = -(1 + \sqrt{5})/2$ and $x_n = x_m$, where $n < m$. Then $F_{n-2}x_1 + F_{n-1}x_2 = F_{m-2}x_1 + F_{m-1}x_2$, and so $(F_{n-2} - F_{m-2})x_1 = (F_{m-1} - F_{n-1})x_2$. Because either $F_{n-2} - F_{m-2}$ or $F_{m-1} - F_{n-1}$ is nonzero, either x_1/x_2 or x_2/x_1 is ratioinal. However, the number $-(1 + \sqrt{5})/2$ is irrational. This is a contradiction. Therefore there is no repeated number. $\square$