Limit of a sequence

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POW2013-16. *For real numbers a, b, find the following limit.*

$$\lim_{n\to\infty} n\left(1-\frac{a}{n}-\frac{b\log(n+1)}{n}\right)^n.$$

Solution. Taking logarithm to the given expression, we obtain by the series expansion for sufficiently large n,

$$\begin{aligned}&\ln n + n\ln\left(1-\frac{a+b\ln(n+1)}{n}\right)\\ &\quad = \ln\frac{n}{n+1} - a + (1-b)\ln(n+1) - \sum_{k=2}^{\infty}\frac{(a+b\ln(n+1))^k}{kn^{k-1}} \qquad (\star)\end{aligned}$$

because $\lim_{n\to\infty}\frac{a+b\ln(n+1)}{n}=0$.

Note also that $\lim_{n\to\infty}\frac{(a+b\ln(n+1))^2}{n}=0$, thus

$$\begin{aligned}\left|\sum_{k=2}^{\infty}\frac{(a+b\ln(n+1))^k}{kn^{k-1}}\right| &\le \sum_{k=2}^{\infty}\left|\frac{(a+b\ln(n+1))^k}{kn^{k-1}}\right|\\ &\le \frac{(a+b\ln(n+1))^2}{n}\sum_{k=0}^{\infty}\left|\frac{a+b\ln(n+1)}{n}\right|^k = o(1).\end{aligned}$$

Therefore, there are three cases.

1. If $b=1$, $\lim_{n\to\infty}(\star)=-a \Rightarrow \lim_{n\to\infty} n\left(1-\frac{a}{n}-\frac{b\log(n+1)}{n}\right)^n = e^{-a}$.

2. If $b<1$, $\lim_{n\to\infty}(\star)=\infty \Rightarrow \lim_{n\to\infty} n\left(1-\frac{a}{n}-\frac{b\log(n+1)}{n}\right)^n = \infty$.

3. If $b>1$, $\lim_{n\to\infty}(\star)=-\infty \Rightarrow \lim_{n\to\infty} n\left(1-\frac{a}{n}-\frac{b\log(n+1)}{n}\right)^n = 0$.

$\square$