POW 2013-14

Nilpotent matrix

수리과학과 김호진

Pow 2013-14 Let A, B are $N \times N$ complex matrices satisfying $\mathrm{rank}(AB - BA) = 1$. Prove that $(AB - BA)^2 = 0$.

Proof. Note that the statement is vacuously true when $N = 1$; set of 1×1 matrices are commutative under matrix multiplication, thus $AB - BA = 0$ for any $A, B \in \mathrm{M}_{1\times 1}(\mathbb{C})$, so $\mathrm{rank}(AB - BA) = 0 \neq 1$. So we assume $N > 1$.

Let $M \in \mathrm{M}_{N\times N}(\mathbb{C})$ be a matrix with $\mathrm{tr}(M) = 0$ and $\mathrm{rank}(M) = 1$. (Such matrices exist when $N > 1$.) Column space of M has dimension 1, *i.e.* there exists a nonzero (column) vector $\mathbf{v} \in \mathbb{C}^N$ such that, for any vector $\mathbf{x} \in \mathbb{C}^N$ there exist $k \in \mathbb{C}$ wich satisfies $M\mathbf{x} = k\mathbf{v}$. Let $\mathbf{v} = (v_1, \cdots, v_N)^T$ for $v_i \in \mathbb{C}$.
Let $M\mathbf{e}_1 = k_1\mathbf{v}, \cdots, M\mathbf{e}_N = k_N\mathbf{v}$. ($k_i \in \mathbb{C}$.) Note that $\mathrm{tr}(M) = \sum_{i=1}^N v_i k_k$. We have

$$\begin{aligned} M^2\mathbf{e}_j &= M(k_j\mathbf{v}) = k_j M \sum_{i=1}^N v_i\mathbf{e}_i \\ &= k_j \sum_{i=1}^N v_i M\mathbf{e}_i = k_j \left(\sum_{i=1}^N v_i k_i\right)\mathbf{v} = k_j\,\mathrm{tr}(M)\mathbf{v} = 0 \end{aligned}$$

for $j = 1, \cdots, N$, *i.e.* $M^2 = O$ a zero matrix. Consquently, If M is rank 1 matrix with zero trace, then $M^2 = O$.

In particular, since $\mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$, $(AB - BA)^2 = O$. ($\mathrm{tr}(AB) = \mathrm{tr}(BA)$ is well know fact, and is immediate from the definition of the matrix multiplication.)

□