POW 2013-13 Functional equation

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Let $h(x) = x^2 + \frac{x}{3} + \frac{1}{9}$. Then h has following properties :

Properties.

(i) $h(x) > 0$ for all $x \in \mathbb{R}$.
(ii) $h(x) \geq x$ for all $x \in \mathbb{R}$. Moreover, $h(x) = x$ has the only solution at $x = 1/3$.
(iii) $h\chi_{[-\frac{1}{6},\infty)}$ defined on $[-1/6, \infty)$ has an inverse function.

Proof.

(i) $h(x) = (x + 1/6)^2 + (1/9 - 1/36) > 0$ for all $x \in \mathbb{R}$.
(ii) $h(x) - x = (x - \frac{1}{3})^2 \geq 0$ for all $x \in \mathbb{R}$.
(iii) An equality $h(x) = (x + 1/6)^2 + (1/9 - 1/36)$ implies that h is monotone-increasing on $[-1/6, \inf)$.

Our goal : f is a constant function.
For any $x \in \mathbb{R}$, $f(x) = f(h(x))$. Since $h(x) > 0$ for all $x \in \mathbb{R}$, it is enough to show f is constant for $x > 0$.

Case 1. $0 < x \leq 1/3$
Since $h(x) - h(1/3) = h(x) - 1/3 = (x - 1/3)(x + 2/3)$, $h(x) \leq 1/3$ for $0 < x \leq 1/3$. For any $0 < x \leq 1/3$, if we let $a_0 = x$ and $a_{n+1} = h(a_n)$ for $n \geq 0$, the sequence $\{a_n\}$ is (monotone) increasing (since $h(x) \geq x$) and bounded above. (i.e. $a_n \leq 1/3$ for all $n \geq 0$) Hence the limit $\alpha \in \mathbb{R}$ exists.
Because h is continuous, we get the equation $\alpha = \lim_{n\to\infty} a_{n+1} = h(\lim_{n\to\infty} a_n) = h(\alpha)$, so $\alpha = 1/3$. Since $f(x) = f(a_0) = f(a_n)$ for all $n \geq 0$ and f is continuous, $f(x) = f(a_0) = f(\alpha) = f(1/3)$.

Case 2. $1/3 < x$
Note that h has an inverse on $[-1/6, \infty)$. Let $a_0 = x$ and $a_{n+1} = h^{-1}(a_n)$ for $n \geq 0$. Similarly $f(a_0) = f(a_n)$ for all $n \geq 0$. Since $h(x) \geq x$, we get $a_{n+1} < a_n$, hence a_n is (monotone) decreasing. Because $h(x) \geq 1/3$ for $x \geq 1/3$, $h^{-1}(x) \geq 1/3$ for $x \geq 1/3$, so $a_n \geq 1/3$ by a simple induction (on n). So the sequence has a limit $\beta \in \mathbb{R}$, which satisfies $\beta = \lim_{n\to\infty} a_{n+1} = h(\lim_{n\to\infty} a_n) = h(\beta)$. Thus $\beta = 1/3$. By the continuity of f, we get $f(x) = f(a_0) = f(\beta) = f(1/3)$.