POW 2013-10

Mean and variance of random variable

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Pow 2013-10 Let random variables $\{X_r : r \geq 1\}$ be independent and uniformly distributed on $[0, 1]$. Let $0 < x < 1$ and define a random variable

$$N = \min\{n \geq 1 : X_1 + X_2 + \cdots + X_n > x\}.$$

Find the mean and variance of N.

Proof. Random variable N is dependent on x. So let's use $N = N(x)$. First we find the p.m.f.(probability mass function) of $N(x)$, and then calculate the mean and variance of $N(x)$.

Let $Y_n = \sum_{i=1}^{n} X_i$ be a new random variable. Here $P(N(x) = n) = P(Y_{n-1} \leq x < Y_n)$ for positive integer n.
Let's find the p.d.f. of Y_n, say $f_n(y)$. (We may assume $0 \leq y \leq 1$ in this problem.) $Y_1 = X_1$, thus

$$f_1(y) = \begin{cases} 1 & y \in [0,1] \\ 0 & \text{otherwise} \end{cases}.$$

And

$$\begin{aligned} f_{n+1}(y) &= \int_{-\infty}^{\infty} f_n(y-t) f_1(t) dt \\ &= \int_0^1 f_n(y-t) dt = \int_0^y f_n(t) dt, \end{aligned}$$

thus $f_n(y) = \frac{1}{(n-1)!} y^{n-1}$ for $n \geq 1$.

Note that $P(N(x) = 1) = (1 - x)$. For $n \geq 2$,

$$\begin{aligned} P(N(x) = n) &= P(Y_{n-1} \leq x < Y_n) \\ &= P(Y_{n-1} \leq x) - P(Y_n < x) \\ &= \frac{x^{n-1}}{(n-1)!} - \frac{x^n}{n!} = \frac{1}{n!}(n-x)x^{n-1}, \end{aligned}$$

which is consistent with $n = 1$ case. So, for $n \in \mathbb{N}$,

$$P(N = n) = \frac{1}{n!}(n - x)x^{n-1}.$$

This is the explicit form of the p.m.f. of $N(x)$.

Now for the mean of $N(x)$, $\mathrm{E}(N(x)) = \sum_{n=1}^{\infty} n \cdot \frac{x^{n-1}(n-x)}{n!}$. This series can be simplified into e^x.

$$\begin{aligned}
\sum_{n=1}^{\infty} n \cdot \frac{x^{n-1}(n-x)}{n!} &= \sum_{n=1}^{\infty} \frac{x^{n-1}(n-x)}{(n-1)!} \\
&= \sum_{n=0}^{\infty} \left(\frac{n}{n!} x^n + (1-x) \cdot \frac{1}{n!} x^n \right) \\
&= xe^x + (1-x)e^x = e^x.
\end{aligned}$$

Thus $\mathrm{E}(N(x)) = e^x$.

And, $\mathrm{Var}(N(x)) = \mathrm{E}(N(x)^2) - \mathrm{E}(N(x))^2$.

$$\begin{aligned}
E(N(x)^2) &= \sum_{n=1}^{\infty} n^2 \cdot \frac{x^{n-1}(n-x)}{n!} \\
&= \sum_{n=1}^{\infty} \frac{n^2 x^{n-1} - n x^n}{(n-1)!} \\
&= \sum_{n=0}^{\infty} \frac{(n+1)^2 x^n - (n+1) x^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} \left(\frac{n^2}{n!} x^n + (2-x) \cdot \frac{n}{n!} x^n + (1-x) \cdot \frac{1}{n!} x^n \right) \\
&= \big(x(x+1) + x(2-x) + (1-x) \big) e^x = (2x+1)e^x,
\end{aligned}$$

thus $\mathrm{Var}(N(x)) = e^x(2x + 1 - e^x)$.
(Here I used $\sum_{n=0}^{\infty} \frac{n}{n!} x^n = xe^x$ and $\sum_{n=0}^{\infty} \frac{n^2}{n!} x^n = x(x+1)e^x$, which can be shown easily.) $\square$