KAIST POW 2013-09.
Inequality for a sequence.

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[PROBLEM]

Let $N > 1000$ be an integer. Define a sequence A_n by

$A_0 = 1$, $A_1 = 0$, $A_{2k+1} = \frac{2k}{2k+1}A_{2k} + \frac{1}{2k+1}A_{2k-1}$, $A_{2k} = \frac{2k-1}{2k}\frac{A_{2k-1}}{N} + \frac{1}{2k}A_{2k-2}$.

Show that the following inequality holds for any integer k with $1 \le k \le \left(\frac{1}{2}\right)N^{\frac{1}{3}}$.

$$A_{2k-2} \le \frac{1}{\sqrt{(2k-2)!}}.$$

[SOLUTION]

We begin with simple lemma.

Lemma : If $N > 1000$, then $\left(\frac{1}{2n} + \frac{n}{N}\right) \le \frac{1}{\sqrt{2n(2n-1)}}$ for all $2 \le n \le \left(\frac{1}{2}\right)N^{\frac{1}{3}} - 1$.

PF) If $\left(1-\frac{1}{2n}\right)\left(1+\frac{2n^2}{N}\right)^2 \le 1$ and $n \in \mathbb{N}$, then $\sqrt{1-\frac{1}{2n}}\left(1+\frac{2n^2}{N}\right) \le 1$ or $\frac{1}{2n}\left(1+\frac{2n^2}{N}\right) \le \frac{1}{\sqrt{2n(2n-1)}}$.

Thus, we will show that $\left(1-\frac{1}{2n}\right)\left(1+\frac{2n^2}{N}\right)^2 \le 1$ for all $2 \le n \le \left(\frac{1}{2}\right)N^{\frac{1}{3}} - 1$.

Let $f(x) = \left(1-\frac{1}{2x}\right)\left(1+\frac{2x^2}{N}\right)^2$, then

$$f\left(\left(\frac{1}{2}\right)N^{\frac{1}{3}} - 1\right) = \left(1 - \frac{1}{N^{\frac{1}{3}} - 2}\right)\left(1 + \frac{2\left\{\left(\frac{1}{2}\right)N^{\frac{1}{3}} - 1\right\}^2}{N}\right)^2$$
$$\le \left(1 - \frac{N^{\frac{2}{3}}}{N}\right)\left(1 + \frac{N^{\frac{2}{3}}}{2N}\right)^2$$
$$= \left(1 - \frac{N^{\frac{2}{3}}}{N}\right)\left(1 + \frac{N^{\frac{2}{3}}}{N} + \frac{N^{\frac{4}{3}}}{4N^2}\right)$$
$$= 1 - \frac{3N^{\frac{1}{3}} + 1}{4N} < 1.$$

Since f is increasing on $(1, \infty)$, $f(n) \le 1$ for all $2 \le n \le \left(\frac{1}{2}\right)N^{\frac{1}{3}} - 1$.

Which finishes the proof of lemma. □

Now, we go on to the main problem.

From $A_1 = 0$, $A_{2k+1} = \frac{2k}{2k+1}A_{2k} + \frac{1}{2k+1}A_{2k-1}$ and $A_{2k} = \frac{2k-1}{2k}\frac{A_{2k-1}}{N} + \frac{1}{2k}A_{2k-2}$, we have

$$A_{2k} = \frac{1}{2k}A_{2k-2} + \frac{2k-1}{2k}\frac{2k-2}{2k-1}\frac{1}{N}A_{2k-2} + \frac{2k-1}{2k}\frac{1}{2k-1}\frac{1}{N}A_{2k-3}$$
$$= \frac{1}{2k}A_{2k-2} + \frac{2k-1}{2k}\frac{2k-2}{2k-1}\frac{1}{N}A_{2k-2} + \frac{2k-1}{2k}\frac{1}{2k-1}\frac{2k-4}{2k-3}\frac{1}{N}A_{2k-4} + \frac{2k-1}{2k}\frac{1}{2k-1}\frac{1}{2k-3}\frac{1}{N}A_{2k-5}$$
$$\vdots$$
$$= \frac{1}{2k}A_{2k-2} + \frac{2k-1}{2k}\frac{2k-2}{2k-1}\frac{1}{N}A_{2k-2} + \frac{2k-1}{2k}\frac{1}{2k-1}\frac{2k-4}{2k-3}\frac{1}{N}A_{2k-4} + \cdots + \frac{2k-1}{2k}\frac{1}{2k-1}\cdots\frac{1}{5}\frac{2}{3}\frac{1}{N}A_2$$

for all $k \in \mathbb{N} - \{1\}$.

Now, we will proof the main result by using Strong Induction.

$A_0 = 1 \le \frac{1}{\sqrt{0!}}$, $A_2 = \frac{1}{2} \le \frac{1}{\sqrt{2!}}$. $(k = 1, 2)$.

Let assume that there exist a natural number n such that $2 \le n \le \left(\frac{1}{2}\right)N^{\frac{1}{3}} - 1$ and

$A_{2m-2} \le \frac{1}{\sqrt{(2m-2)!}}$ for all $1 \le m \le n$. $(k \le n)$.

Since $A_{2n} = \frac{1}{2n}A_{2n-2} + \frac{2n-1}{2n}\frac{2n-2}{2n-1}\frac{1}{N}A_{2k-2} + \frac{2n-1}{2n}\frac{1}{2n-1}\frac{2n-4}{2n-3}\frac{1}{N}A_{2n-4} + \cdots + \frac{2n-1}{2n}\frac{1}{2n-1}\cdots\frac{1}{5}\frac{2}{3}\frac{1}{N}A_2$

and
$$\frac{1}{(2n-1)\cdots(2n-2p+1)}A_{2n-2p-2} \le \frac{1}{(2n-1)\cdots(2n-2p+1)}\frac{1}{\sqrt{(2n-2p-2)!}}$$
$$\le \frac{1}{\sqrt{(2n-2)(2n-3)}\cdots\sqrt{(2n-2p)(2n-2p-1)}}\frac{1}{\sqrt{(2n-2p-2)!}}$$
$$= \frac{1}{\sqrt{(2n-2)!}},$$

$$A_{2n} \le \frac{1}{\sqrt{(2n-2)!}}\left(\frac{1}{2n} + \frac{1}{N}\left(\frac{2n-2}{2n} + \frac{2n-1}{2n}\frac{2n-4}{2n-3} + \frac{2n-1}{2n}\frac{2n-6}{2n-5} + \cdots + \frac{2n-1}{2n}\frac{2}{3}\right)\right)$$

Hence $A_{2n} \le \frac{1}{\sqrt{(2n-2)!}}\left(\frac{1}{2n} + \frac{n}{N}\right)$.

(By our lemma) $\le \frac{1}{\sqrt{(2n)!}}$. $(k = n+1)$.

Hence $A_{2k-2} \le \frac{1}{\sqrt{(2k-2)!}}$ for any integer k with $1 \le k \le \left(\frac{1}{2}\right)N^{\frac{1}{3}}$.

Which is what we wanted. ■