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1. Problem

Consider the unit sphere in R^n. Find the maximum number of points on the sphere such that the (Euclidean) distance between any two of these points is larger than $\sqrt{2}$.

2. Solution

For $n \geq 1$, let S_{n-1} be $(n-1)$-dimensional unit sphere in R^n. That is, $S_{n-1} = \{x \in \mathrm{R}^n | |x| = 1\}$.

If A is a set of points on S_{n-1} such that the (Euclidean) distance between any two of these points is larger than $\sqrt{2}$, then A should be finite: since S_{n-1} is compact, if A is infinite, it should have arbitrarily close two points, and clearly they cannot have distance larger than $\sqrt{2}$.

Then defining M_n to be the maximum number of points on S_{n-1} such that distance between any of two of these points is larger than $\sqrt{2}$ makes sense and M_n should also be finite.

We begin with 2 simple lemmas. The first lemma states that distance between two points X, Y on the sphere is larger than $\sqrt{2}$ if and only if the angle $\angle XOY$ is obtuse, where O is the origin.

Lemma 1. Let $x, y \in S_{n-1}$. Then $d(x, y) > \sqrt{2}$ if and only if $x \cdot y < 0$.

Proof)

$d(x, y) > \sqrt{2}$ if and only if $d(x, y)^2 > 2$. But $d(x, y)^2 = |x - y|^2 = |x|^2 + |y|^2 - 2x \cdot y = 2 - 2x \cdot y$. Therefore, $d(x, y)^2 > 2$ if and only if $x \cdot y < 0$.
$\square$

The next lemma states that if three points X, Y, Z on the sphere satisfy $\angle XOY, \angle XOZ, \angle YOZ$ are all obtuse, then if we project Y, Z to the hyperplane plane passing through O and orthogonal to OX and letting the obtained points Y', Z', then $\angle Y'OZ'$ is again obtuse.

Lemma 2. Let $x, y, z \in S_{n-1}$ with $x \cdot y < 0, y \cdot z < 0, z \cdot x < 0$. Then $\mathrm{proj}_{x^\perp} y \cdot \mathrm{proj}_{x^\perp} z < 0$.

Note that the hypotheses of the above lemma implies that any two of them cannot be antipodes, i. e. $x = -y$ or $y = -z$ or $z = -x$ cannot be possible.

Proof of Lemma 2)

Note that $y = \mathrm{proj}_{x^\perp} y + \mathrm{proj}_x y = \mathrm{proj}_{x^\perp} y + (x \cdot y)x$ and similarly $z = \mathrm{proj}_{x^\perp} z + \mathrm{proj}_x z = \mathrm{proj}_{x^\perp} z + (x \cdot z)x$.

Then $\mathrm{proj}_{x^\perp} y \cdot \mathrm{proj}_{x^\perp} z = (y - (x \cdot y)x) \cdot (z - (x \cdot z)x)$
$= y \cdot z - (x \cdot z)(x \cdot y) < 0$, as desired. $\square$

Now we go on to the main result. First we note that For $n = 1$ $M_n = 2$. Now let $n > 1$. One obvious note is that $M_n \geq 1$ for all $n \geq 1$.

Let A_n be a set of points in S_{n-1} such that distance between any points is larger than $sqrt2$, and having M_n points.

Then since $|A_n| \geq 1$ for $n \geq 1$, we can choose x from A_n.

Then by the virtue of Lemma 2, $B = \{\frac{\mathrm{proj}_{x^\perp} y}{|\mathrm{proj}_{x^\perp} y|} | y \in A_n - \{x\}\}$ is a set of points on the unit sphere $S_{n-1} \cap x^\perp = S_{n-2}$ (where $x^\perp$ is identified with R^{n-1}) such that distance between any of two of these points is larger than $\sqrt{2}$.

Then $M_n = |B| + 1 \leq M_{n-1} + 1$.

On the other hand, we can identify R^n with n-dimensional subspace of R^{n+1} by identifying points x of R^n with $(x, 0) \in \mathrm{R}^{n+1}$. Then $S_{n-1} = S_n \cap e_{n+1}^\perp \in S_n$.

Now since A_n is a finite set, we can choose sufficiently small $\epsilon > 0$ such that $\{(\sqrt{1-\epsilon^2}x, -\epsilon) | x \in A_n\} \subseteq S_n$ and $d((\sqrt{1-\epsilon^2}x, -\epsilon), (\sqrt{1-\epsilon^2}y, -\epsilon)) = \sqrt{(1-\epsilon^2)d(x,y)^2 + \epsilon^2} > \sqrt{2}$.

Then $\{(\sqrt{1-\epsilon^2}x, -\epsilon) | x \in A_n\} \cup \{e_{n+1}\} \subseteq S_n$ and distance between any two of these points is larger than $\sqrt{2}$. Therefore, $M_n + 1 = |A_n| + 1 \leq M_{n+1}$.

Therefore, $M_{n-1} + 1 = M_n$ for $n \geq 1$ and $M_1 = 2$ gives $M_n = n + 1$ for all $n \geq 1$.