POW 2013-05 Solution 20130716 황성호 (무학과)

1. Problem

Prove that F has at least 80 zeros in the interval (0,2013). $F(x)=\sum_{n=1}^{1000}\cos(n^{1.5}x)$.

2. Lemma

For given continuous and differentiable function $F(x)$, define $G(x)=\int_0^x F(t)dt$.

(i) $G(x)$ is also a continuous and differentiable function.

(ii) If $G(x)$ has at least m zeros, then $F(x)$ has at least $m-1$ zeros each in the interval of consecutive two zeros of $G(x)$.

3. Proof of Lemma

(i) Trivial.

(ii) Suppose that m zeros of $G(x)$ are $x_1, x_2, \cdots, x_m$ with $x_1 < x_2 < \cdots < x_m$.

Since $F(x)$ is continuous and differentiable function, we can apply the 'Mean Value Theorem'. Then, there exists at least one zero of $F(x)$ in the interval (x_i, x_{i+1}).

Then, $F(x)$ has at least $m-1$ zeros $y_1, y_2, \cdots, y_{m-1}$ with
$x_1 < y_1 < x_2 < y_2 < \cdots y_{m-1} < x_m$.

4. Proof of Problem

Define $G(x)=\int_0^x F(t)dt=\sum_{n=1}^{1000} n^{-1.5}\sin(n^{1.5}x)$.

Define $H(x)=\int_0^x G(t)dt-\sum_{n=1}^{1000} n^{-3}\cos(n^{1.5}0)=-\sum_{n=1}^{1000} n^{-3}\cos(n^{1.5}x)$.

($\sum_{n=1}^{1000} n^{-3}\cos(n^{1.5}0)$ is just a constant term.)

$H(x)=-\sum_{n=1}^{1000} n^{-3}\cos(n^{1.5}x)=-\cos x-\sum_{n=2}^{1000} n^{-3}\cos(n^{1.5}x)$

$|\sum_{n=2}^{1000} n^{-3}\cos(n^{1.5}x)| \le \sum_{n=2}^{1000} n^{-3} < 1$

It means that $-\cos x-1 < H(x) < -\cos x+1$.

Since $H(2k\pi) < -\cos(2k\pi)+1=0$, $H(2k\pi+\pi) > -\cos(2k\pi+\pi)-1=0$, there exist at least one zero of $H(x)$ in the interval $(2k\pi, 2k\pi+\pi)$ for $\forall k \in Z$.

$\therefore H(x)$ has at least $[\frac{2013}{2\pi}]=320$

Since $F(x)$, $G(x)$, $H(x)$ are all continuous, differentiable, $G(x)$ has at least 319 zeros and $F(x)$ has at least 318 zeros.

$\therefore$ Consequently, $F(x)$ has at least 80 zeros in the interval (0,2013).