(Solution of POW 2013-04)

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Let's use induction on $N \geq 2$.

(i) $N=2$

Let $H=\begin{pmatrix} a & h \\ h & b \end{pmatrix}$ where $|a|$, $|b|<1$ and $|h|>4$.

Then, form the eigenvalue equation, $p(\lambda)=\det(H-\lambda I)=0$, we have

$$p(\lambda)=(a-\lambda)(b-\lambda)-h^2=\lambda^2-(a+b)\lambda+(ab-h^2)=0$$

$$\Rightarrow \lambda^H_{\max}=\frac{a+b}{2}+\sqrt{\frac{(a-b)^2}{4}+h^2}>|h|-\frac{|a|+|b|}{2}>3.$$

(ii) Assume that the statement is true for $N=n-1$ for some $n \geq 3$.

(iii) Now it is enough to show the statement is true for $N=n$.

Let H be an $n\times n$ real symmetric matrix with $|H_{mm}|<1$ for $1\leq m\leq n$ and $\exists i,j$ s.t. $|H_{ij}|>4$.

Then $\exists k\in\{1,\cdots n\}$ s.t. $i\neq k\neq j$ and we can obtain an $(n-1)\times(n-1)$ real symmetric matrix H' from H by deleting the kth row and the kth column of H, which satisfies $|H'_{mm}|<1$ for $1\leq m\leq n-1$ and $\exists i',j'$ s.t. $|H'_{i'j'}|>4$.

Then by (ii), $\lambda^{H'}_{\max}>3$.

Note that for any real symmetric matrix A,

$$\lambda^A_{\max}=\max_{x\neq 0}\frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}x}. \qquad \cdots\cdots\cdots\cdots(1)$$

Let y be a vector in $\mathbb{R}^n$ whose kth entry is zero and $y'\in\mathbb{R}^{n-1}$ be a vector obtained by deleting the kth entry of y. Then by (1),

$$\lambda^H_{\max}=\max_{x\neq 0}\frac{x^{\mathrm{T}}Hx}{x^{\mathrm{T}}x}\geq\max_{y\neq 0}\frac{y^{\mathrm{T}}Hy}{y^{\mathrm{T}}y}.$$

Since $y^{\mathrm{T}}Hy=(y')^{\mathrm{T}}H'(y')$ and $y^{\mathrm{T}}y=(y')^{\mathrm{T}}y'$ for $(y)_k=0$, we have

$$\lambda^H_{\max}\geq\max_{y'\neq 0}\frac{y'^{\mathrm{T}}H'y'}{y'^{\mathrm{T}}y'}=\lambda^{H'}_{\max}>3.$$

This completes the proof.