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Functional equation

수리과학과 김호진

Pow 2013-02 Let $\mathbb{Z}^+$ be the set of positive integers. Suppose that $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfies the following conditions.

i) $f(f(x)) = 5x$

ii) If $m \geq n$, then $f(m) \geq f(n)$

iii) $f(1) \neq 2$

find $f(256)$.

solution. $f(256) = 506$.
If $f(m) = f(n)$, then $f(f(m)) = 5m = f(f(n)) = 5n$ so $m = n$. Therefore f is injective. Thus, f is *strictly* increasing, *i.e.* $m > n$ implies $f(m) > f(n)$.
$f(1) = 3$. One can deduce this by elminating every other possibility.
If $f(1) = 1$, then $5 = f(f(1)) = f(1) = 1$, which is contradiction. So $f(1) \neq 1$. And $f(1) \neq 2$ is given.
If $f(1) = 4$, then $f(f(1)) = f(4) = 5$. Thus $f(1) = 4 < f(2) < f(4) = 5$. But there is no integer between 4 and 5. So $f(1) \neq 4$.
If $f(1) \geq 5$, then $f(f(1)) = 5 \leq f(1)$. But $f(1) > 1$ and f is strictly increasing, so it is impossible. So $f(1) < 5$.
Therefore $f(1) = 3$. And thus $f(3) = f(f(1)) = 5$.
Let me introduce a simple lemma.

Lemma 1. For $n \geq 0$, $f(5^n) = 3 \cdot 5^n$ and $f(3 \cdot 5^n) = 5^{n+1}$.

proof of Lemma 1. For $n = 0$, we have $f(1) = 3$ and $f(3) = 5$.
If $f(5^k) = 3 \cdot 5^k$ and $f(3 \cdot 5^k) = 5^{k+1}$, then $f(5^{k+1}) = f(f(3 \cdot 5^k)) = 3 \cdot 5^{k+1}$ and thus $f(3 \cdot 5^{k+1}) = f(f(5^{k+1})) = 5^{k+2}$. So by induction, the lemma is true. □

Note that $f(3 \cdot 5^n) - f(5^n) = 3 \cdot 5^n - 5^n$. Since f is strictly increasing, if $5^n \leq m < 3 \cdot 5^n$ for some $n \geq 0$, then $f(m+1) = f(m) + 1$, and thus $f(m) = f(5^n) + (m - 5^n)$.
$5^3 = 125 \leq 256 < 3 \cdot 5^3 = 375$. so, $f(256) = f(5^3) + (256 - 125) = 375 + 131 = 506$. □

Note. One can determine the value of $f(n)$ for all $n \in \mathbb{Z}^+$.
Here let $[a, b)_{\mathbb{Z}^+} = \{x \in \mathbb{Z}^+ : a \leq x < b\} = \mathbb{Z}^+ \cap [a, b)$. $\mathbb{Z}^+$ is partitioned into

$$\mathbb{Z}^+ = \bigsqcup_{n \geq 0} \left([5^n, 3 \cdot 5^n)_{\mathbb{Z}^+} \sqcup [3 \cdot 5^n, 5^{n+1})_{\mathbb{Z}^+} \right).$$

If $m \in [5^n, 3 \cdot 5^n)_{\mathbb{Z}^+}$ for some $n \geq 0$, then $f(m) = 3 \cdot 5^n + (m - 5^n) = m + 2 \cdot 5^n$.
If $m \in [3 \cdot 5^n, 5^{n+1})_{\mathbb{Z}^+}$ for some $n \geq 0$, $m - 2 \cdot 5^n \in [5^n, 3 \cdot 5^n)_{\mathbb{Z}^+}$, and we have $f(m - 2 \cdot 5^n) = m$. Thus, $f(m) = f(f(m - 2 \cdot 5^n)) = 5m - 2 \cdot 5^{n+1}$. Consequently,

$$f(m) = \begin{cases} m + 2 \cdot 5^n & \text{if } m \in [5^n, 3 \cdot 5^n)_{\mathbb{Z}^+} \text{ for some } n \geq 0 \\ 5m - 2 \cdot 5^{n+1} & \text{if } m \in [3 \cdot 5^n, 5^{n+1})_{\mathbb{Z}^+} \text{ for some } n \geq 0 \end{cases}.$$