POW 2013-01 Solution 20080286 Joonhyun La (Mathematical Science)

1. Problem

Let A, B be $N \times N$ symmetric matrices with eigenvalues $\lambda_1^A \le \lambda_2^A \le \cdots \le \lambda_N^A$ and $\lambda_1^B \le \lambda_2^B \le \cdots \le \lambda_N^B$. Prove that

$$\sum_{i=1}^{N} |\lambda_i^A - \lambda_i^B|^2 \le tr(A-B)^2$$

2. Solution

First, we may assume that both λ_1^A and λ_1^B are nonnegative; for any t if we let $C = A + tI$ and $D = B + tI$, where I is the $N \times N$ identity matrix, then C and D are $N \times N$ symmetric matrices with eigenvalues $\lambda_i^C = \lambda_i^A + t$ and $\lambda_i^D = \lambda_i^B + t$ for $i = 1, 2, \cdots, N$. Thus, $\lambda_i^C - \lambda_i^D = \lambda_i^A - \lambda_i^B$ and $C - D = A - B$, so the values of left-handed side and right-handed side of the desired inequality remain unchanged if we substitute C and D instead of A and B, respectively. Taking sufficiently large t, we may assume that all eigenvalues of C and D are nonnegative.

Next, we may assume that A may be considered as a diagonal one, with $A_{ii} = \lambda_i^A$. This is because of the fact that the left-handed side and right-handed side of the desired inequality are invariant under similarity transformation, and since A is symmetric $A = P^T D P$ for some orthogonal matrix P, and a diagonal matrix D with $D_{ii} = \lambda_i^A$.

Therefore we may assume that $A_{ij} = \lambda_i^A$ if $i = j$ and 0 if $i \ne j$. Then since B is also a symmetric matrix, $B = P^T D_B P$ for some orthogonal matrix $P = (e_{ij})$ and a diagonal matrix D_B with $D_{Bii} = \lambda_i^B$. Computing the entries of B, we get

$$B_{ij} = \sum_{k=1}^{N} \lambda_k^B e_{ki} e_{kj} \qquad \text{..................(1)}$$

Since $A - B$ is also symmetric, we have

$$tr(A-B)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} {(A-B)_{ij}}^2 \qquad \text{..................(2)}$$

Substituting (1) to (2), we get

$$tr(A-B)^2 = \sum_{i=1}^{N} (\lambda_i^A - \sum_{k=1}^{N} \lambda_k^B e_{ki}^2)^2 + \sum_{i=1}^{N} \sum_{j=1, j\ne i}^{N} (\sum_{k=1}^{N} \lambda_k^B e_{ki} e_{kj})^2$$

$$= \sum_{i=1}^{N} (\lambda_i^A)^2 - 2 \sum_{i=1}^{N} \lambda_i^A (\sum_{k=1}^{N} \lambda_k^B {e_{ki}}^2) + \sum_{i=1}^{N} \sum_{j=1}^{N} (\sum_{k=1}^{N} \lambda_k^B e_{ki} e_{kj})^2$$

But

$$\sum_{i=1}^{N}\sum_{j=1}^{N}(\sum_{k=1}^{N}\lambda_k^B e_{ki}e_{kj})^2 = \sum_{k=1}^{N}\sum_{l=1}^{N}\lambda_k^B\lambda_l^B(\sum_{i=1}^{N}\sum_{j=1}^{N}e_{ki}e_{kj}e_{li}e_{lj})$$
$$= \sum_{k=1}^{N}\sum_{l=1}^{N}\lambda_k^B\lambda_l^B\sum_{i=1}^{N}(e_{ki}e_{li})\sum_{j=1}^{N}(e_{kj}e_{lj}) = \sum_{i=1}^{N}(\lambda_i^B)^2$$

Since

$$\sum_{i=1}^{N}(e_{ki}e_{li}) =< (e_{k1}, e_{k2}, \cdots, e_{kN}), (e_{l1}, e_{l2}, \cdots, e_{lN}) >= \delta_{kl}$$

Where δ_{kl} equals 1 if $k = l$ and equals 0 if $k \neq l$, due to orthogonality of $P = (e_{ij})$.

Consequently,

$$tr(A-B)^2 = \sum_{i=1}^{N}((\lambda_i^A)^2 + (\lambda_i^B)^2) - 2\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i^A\lambda_j^B(e_{ji})^2$$
$$= \sum_{i=1}^{N}(\lambda_i^A - \lambda_i^B)^2 + 2(\sum_{i=1}^{N}\lambda_i^A\lambda_i^B - \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i^A\lambda_j^B(e_{ji})^2)$$

And it suffices to show that

$$\sum_{i=1}^{N}\lambda_i^A\lambda_i^B - \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i^A\lambda_j^B(e_{ji})^2 \geq 0 \qquad \text{.........(3)}$$

Inequality (3) immediately follows from the following lemma;

Lemma Let $0 \leq x_1 \leq \cdots \leq x_N$, $0 \leq y_1 \leq \cdots \leq y_N$ and $A = (a_{ij})$ be an $N \times N$ matrix with $a_{ij} \geq 0$ for all i and j, and $\sum_{i=1}^{N} a_{ij} = 1$ for all j and $\sum_{j=1}^{N} a_{ij} = 1$ for all i. Then

$$\sum_{i=1}^{N}x_iy_i \geq \sum_{i=1}^{N}\sum_{j=1}^{N}x_ia_{ij}y_j$$

Proof of the Lemma) I have referred [M].

Let $z_1 = x_1$, $z_i = x_i - x_{i-1}$ for $i \geq 2$ and $w_1 = y_1$, $w_j = y_j - y_{j-1}$ for $j \geq 2$.

Then

$$\sum_{i=1}^{N}x_iy_i - \sum_{i=1}^{N}\sum_{j=1}^{N}x_ia_{ij}y_j = \sum_{i=1}^{N}\sum_{j=1}^{N}(\delta_{ij} - a_{ij})x_iy_j$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{N}(\delta_{ij} - a_{ij})(\sum_{r=1}^{i}z_r)(\sum_{s=1}^{j}w_s)$$
$$= \sum_{r=1}^{N}\sum_{s=1}^{N}z_rw_s(\sum_{i=r}^{N}\sum_{j=s}^{N}(\delta_{ij} - a_{ij}))$$

And it suffices to show that $\sum_{i=r}^{N}\sum_{j=s}^{N}(\delta_{ij} - a_{ij}) \geq 0$ for every r and s.

If $r \geq s$, then

$$\sum_{i=r}^{N}\sum_{j=s}^{N}(\delta_{ij} - a_{ij}) = (N-r+1) - \sum_{i=r}^{N}\sum_{j=s}^{N} a_{ij} \geq (N-r+1) - \sum_{i=r}^{N}\sum_{j=1}^{N} a_i j = 0$$

And similar argument works for $r \leq s$ case.

3. Reference

[M] Mirsky L., *A trace inequality of John von Neuman*, Monatshefte f¨ur Math. **79** (1975), 303–306.