POW2012-23 2012 엄태현

Let n combination m as $C_{n,m}$

Let $x_{n,m}=\frac{1}{4^{n-1}}(2n-2m+1)(2m-1)C_{2(n-m),n-m}C_{2(m-1),m-1}$ for $m=1,2,3,\cdots,n$

Then $x_{n+1,m}=(1+\frac{1}{2(n-m+1)})x_{n,m}$, $x_{n+1,m+1}=(1+\frac{1}{2m})x_{n,m}$, $x_{n,m}=x_{n,n+1-m}$

Then by Comparing coefficient of $\frac{2\times 4^2}{(1-4x)^3}=(\frac{1}{1-4x})''=2\times 4^2((1-4x)^{-\frac{3}{2}})^2$

$$\sum_{i=1}^{n}x_{n,i}=\frac{n(n+1)}{2}$$

We can easily prove that this equation has unique solution by Gauss elimination.

Now when $n=1$, $x_{1,1}$ is a solution

For assumption let $x_{k,1},x_{k,2},x_{k,2},\cdots,x_{k,k}$ is solution when $n=k$

$$\sum_{j=1}^{k+1}\frac{x_{k+1,j}}{1-4(i-j)^2}-\sum_{j=1}^{k}\frac{x_{k,j}}{1-4(i-j)^2}=\frac{x_{k+1,k+1}}{1-4(k+1-i)^2}+\sum_{j=1}^{k}\frac{x_{k,j}}{(2k-2j+2)(1-2i+2j)(1+2i-2j)}$$

$$\sum_{j=1}^{k+1}\frac{x_{k+1,j}}{1-4(i+1-j)^2}-\sum_{j=1}^{k}\frac{x_{k,j}}{1-4(i-j)^2}=\frac{x_{k+1,1}}{1-4i^2}+\sum_{j=1}^{k}\frac{x_{k,j}}{2j(1-2i+2j)(1+2i-2j)}$$

$$\frac{x_{k,j}}{(2k-2j+2)(1-2i+2j)(1+2i-2j)}$$

$$=\frac{1}{2}(\frac{1}{2(k-i)+3}-\frac{1}{2(k-i)+1})\frac{x_{k,j}}{(2k-2j+2)}+x_{k,j}\frac{4(k-i)-2(2i-2j)+4}{(2(k-i)+3)(2(k-i)+1)(1-4(i-j)^2)})$$

$$=\frac{1}{2}(\frac{-2}{4(k+1-i)^2-1})(x_{k,j+1}-x_{k,j})+\frac{2(k-2i+1+j)}{(2(k-i)+3)(2(k-i)+1)}\frac{x_{k,j}}{1-4(i-j)^2}$$

Thus,

$$\sum_{j=1}^{k+1}\frac{x_{k+1,j}}{1-4(i-j)^2}-\sum_{j=1}^{k}\frac{x_{k,j}}{1-4(i-j)^2}=\frac{1}{1-4(k+1-i)^2}(\sum_{j=1}^{k+1}x_{k+1,j}-\sum_{j=1}^{k}x_{k,j})+\sum_{j=1}^{k}\frac{2(k-2i+1+j)}{(2(k-i)+3)(2(k-i)+1)}\frac{x_{k,j}}{1-4(i-j)^2}$$

$$=\frac{1}{1-4(k+1-i)^2}(\frac{(k+1)(k+2)}{2}-\frac{(k+1)k}{2})+\sum_{j=1}^{k}\frac{2(k-2i+1+j)}{(2(k-i)+3)(2(k-i)+1)}\frac{x_{k,j}}{1-4(i-j)^2}$$

$$=\frac{k+1}{1-4(k+1-i)^2}+\sum_{j=1}^{k}\frac{2(k-2i+1)}{(2(k-i)+3)(2(k-i)+1)}\frac{x_{k,j}}{1-4(i-j)^2}+\sum_{j=1}^{k}\frac{1}{(2(k-i)+3)(2(k-i)+1)}\frac{2jx_{k,j}}{1-4(i-j)^2}$$

$$=\frac{1}{1-4(k+1-i)^2}(k+1-2(k-2i+1)-\sum_{j=1}^{k}\frac{2jx_{k,j}}{1-4(i-j)^2})$$

$$=\frac{1}{1-4(k+1-i)^2}(4i-k-1-\sum_{j=1}^{k}\frac{2jx_{k,j}}{1-4(i-j)^2})$$

Similary,

$$\sum_{j=1}^{k+1}\frac{x_{k+1,j}}{1-4(i+1-j)^2}-\sum_{j=1}^{k}\frac{x_{k,j}}{1-4(i-j)^2}=\frac{1}{1-4i^2}(k+1-4i+\sum_{j=1}^{k}\frac{2jx_{k,j}}{1-4(i-j)^2})$$

Thus,

$$(1-4i^2)(\sum_{j=1}^{k+1}\frac{x_{k+1,j}}{1-4(i+1-j)^2}-1)+(1-4(k+1-i)^2)(\sum_{j=1}^{k+1}\frac{x_{k+1,j}}{1-4(i-j)^2}-1)=0$$

Using $x_{n,m} = x_{n,n+1-m}$, we can prove

$$\sum_{j=1}^{k+1} \frac{x_{k+1,j}}{1-4(i-j)^2} = 1$$

Thus $x_{k+1,1}, x_{k+1,2}, \cdots, x_{k+1,k+1}$ is solution when $n = k+1$

By Mathematical induction, $x_{n,i}$, $i = 1,2,3,\cdots,n$ are unique solution and

$\sum_{i=1}^{n} x_{n,i} = \frac{(n+1)n}{2}$. Thus problem proved