POW 2012-24 Determinant of a Huge Matrix

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Proof.
For a subset sequence $(S_1, S_2, \cdots, S_{2^n-1})$ which S_i is all nonempty distinct subset of $\{1, 2, \cdots, n\}$, we may say $(2^n-1)\times(2^n-1)$ matrix $A = (a_{i,j})$ is "**generated by** $(S_1, S_2, \cdots, S_{2^n-1})$" when $a_{i,j} = 1(S_i \cap S_j \neq \emptyset), 0(\text{otherwise})$.

Claim. Let A, A' be a matrix which are generated by $(S_1, S_2, \cdots, S_{2^n-1})$, $(S'_1, S'_2, \cdots, S'_{2^n-1})$, respectively. Then, $|\det A| = |\det A'|$.

Proof of Claim. It it is enough to show that the statements holds when $S_i = S'_i(i \neq p, q)$ and $S_p = S'_q, S_q = S'_p$. Interchanging rows and columns does not change the size of determinant. So, we interchange pth row and qth rows in A. After that, interchange pth column and qth column. From this, we can obtain A' from A. Therefore, we can conclude that $|\det A| = |\det A'|$.

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Now, we will prove that $|\det A| = 1$ for all n with induction on n. When $n = 1$, it is trivial. Suppose the statements holds when $n = k$. Now let's look when $n = k+1$. Let $(S_1, S_2, \cdots, S_{2^k-1})$ be all nonempty distinct subset of $\{1, 2, \cdots, k\}$.

Let's think $S = (\{k+1\}, S_1 \cup \{k+1\}, \cdots, S_{2^k-1} \cup \{k+1\}, S_1, S_2, \cdots, S_{2^k-1})$. It is all nonempty distinct subset of $\{1, 2, \cdots, k, k+1\}$. By **Claim**, it is enough to calculate the determinant A which is generated by S.

Let's denote the matrix as A_e subtracting 1st row from $2, 3, \cdots, 2^k$th row of A. Let A' be a $(2^k-1)\times(2^k-1)$ matrix which is generated by $(S_1, S_2, \cdots, S_{2^k-1})$. Since $S_i \cap (S_j \cup \{k+1\}) = S_i \cap S_j$ and A_e's first column is $(1, 0, 0, \cdots, 0)$, then it is enough to calculate determinant of the following:

$$\left(\begin{array}{c|c} O & A' \\ \hline A' & A' \end{array}\right)$$

Since determinant of this matrix is $-(\det A')^2 = -1$, then $|\det A| = 1$. By induction, we proved it.

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