KAIST POW 2012-22

KAIST Myeongjae Lee

(Simple integral) Compute $\int_0^1 \frac{x^k-1}{\ln x} dx$

Proof. Define a function $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ by $f(k) = \int_0^1 \frac{x^k-1}{\ln x} dx$. Since $f(0) = 0 < \infty$ and the integral support $[0, 1]$ is compact, $f(k)$ is finite and differentiable near 0. In this neighborhood, and $k > -1$, $f'(k) = \int_0^1 x^k dx = \frac{1}{k+1}$. This implies that for $k > -1$, $f(k)$ is finite and

$$\int_0^1 \frac{x^k - 1}{\ln x} dx = \ln(k + 1) \text{ for } k > -1$$

For $k < -1$, use $\lim_{k \to -1^+} f(k) = -\infty$. Clearly $f(k)$ is decreasing since x^k is decreasing for $x \in [0, 1]$. Therefore,

$$\int_0^1 \frac{x^k - 1}{\ln x} dx = -\infty \text{ for } k \leq -1$$