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(Determinant of a random 0-1 matrix) Let n be a fixed positive integer and let $p \in (0,1)$. Let D_n be the determinant of a random $n \times n$ 0-1 matrix whose entries are independent identical random variables, each of which is 1 with the probability p and 0 with the probability $1-p$. Find the expected value and variance of D_n.

Proof. If x is such a random variable, $E(x) = E(x^2) = p$.

If $n = 1$, $D_n = x$. So $E(D_n) = E(x) = p$, $Var(D_n) = E(x^2) - E(x)^2 = p - p^2$.

Suppose $n \geq 2$. Then $D_n = \sum_{\sigma \in S_n} s(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)} = \sum_{\sigma \in S_n} A_\sigma$ where $A_\sigma = s(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$ and $s(\sigma)$ is a sign of σ.

Since entries are independent, $E(A_\sigma) = s(\sigma)p^n$, and

$$E(D_n) = p^n \sum_{\sigma \in S_n} s(\sigma) = 0$$

Note that $E(A_\sigma A_\tau) = s(\tau^{-1}\sigma)p^{2n-i}$ where $i = |\{j | \sigma(j) = \tau(j)\}|$. If we take $\phi = \tau^{-1}\sigma \in S_n$, this becomes $s(\phi)p^{2n-i_\phi}$ where $i_\phi = |\{j | \phi(j) = j\}|$ and there are $n!$ pair of σ, τ for each ϕ.

Then $Var(D_n) = E(D_n^2) - E(D_n)^2 = E(D_n^2) = \sum_{\sigma,\tau \in S_n} E(A_\sigma A_\tau) = n! \sum_{\phi \in S_n} s(\phi) p^{2n-i_\phi} = n! p^{2n} \sum_{\phi \in S_n} s(\phi) p^{-i_\phi}$. The summation term is clearly the determinant of a matrix with 1/p for all diagonals and 1 for others.

To compute this, subtracting 1st row to other rows,

$$\begin{vmatrix} \frac{1}{p} & 1 & 1 & \cdots & 1 \\ 1-\frac{1}{p} & \frac{1}{p}-1 & 0 & \cdots & 0 \\ 1-\frac{1}{p} & 0 & \frac{1}{p}-1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 1-\frac{1}{p} & 0 & \vdots & & \frac{1}{p}-1 \end{vmatrix}$$

Pulling out $\frac{1}{p} - 1$, and subtracting other rows to 1st row, it becomes

$$\left(\frac{1}{p}-1\right)^{n-1} \begin{vmatrix} \frac{1}{p}+n-1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ -1 & 0 & \vdots & & 1 \end{vmatrix} = n\left(\frac{1}{p}-1\right)^{n-1} + \left(\frac{1}{p}-1\right)^n$$

Therefore, $Var(D_n) = n! p^{2n} \left[n\left(\frac{1}{p}-1\right)^{n-1} + \left(\frac{1}{p}-1\right)^n \right]$.