KAIST POW 2012-19

KAIST Myeongjae Lee

October 16, 2012

(A limit of a sequence involving a square root) Let $a_0 = 3$ and $a_n = a_{n-1} + \sqrt{{a_{n-1}}^2 + 3}$ for all $n \geq 1$. Determine $\lim_{n\to\infty} \frac{a_n}{2^n}$.

Proof. Note that $\cot\theta = \cot 2\theta + \sqrt{\cot^2 2\theta + 1}$ for $0 < \theta < \pi/2$. Since a_n is determined by $a_0 = 3 = \sqrt{3}\cot\pi/6$, we get

$$a_n = \sqrt{3}\cot(\frac{\pi}{6 \times 2^n})$$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{2^n} = \lim_{n\to\infty} \frac{6\sqrt{3}}{\pi} \frac{\pi/(6 \times 2^n)}{\tan(\pi/(6 \times 2^n))} = \frac{6\sqrt{3}}{\pi}$$