Proof. Define $f(z) := \frac{1}{z^2}\tan(\frac{\pi z}{m})\tan(\frac{\pi z}{n})\cot(\pi z)$.
Let us consider a contour C_N which is a square of side $2N + \frac{3}{2}$ centered on the origin. For any point on horizontal sides of contour,

$$\begin{aligned}
|f(z)| &= \left|\frac{1}{z^2} \times \frac{e^{-i\frac{\pi z}{m}} - e^{i\frac{\pi z}{m}}}{e^{-i\frac{\pi z}{m}} + e^{i\frac{\pi z}{m}}} \times \frac{e^{-i\frac{\pi z}{n}} - e^{i\frac{\pi z}{n}}}{e^{-i\frac{\pi z}{n}} + e^{i\frac{\pi z}{n}}} \times \frac{e^{-i\pi z} + e^{i\pi z}}{-e^{-i\pi z} - e^{i\pi z}}\right| \\
&\leq \left|\frac{1}{N^2} \times \frac{|e^{-i\frac{\pi z}{m}}| + |e^{i\frac{\pi z}{m}}|}{|e^{-i\frac{\pi z}{m}}| - |e^{i\frac{\pi z}{m}}|} \times \frac{|e^{-i\frac{\pi z}{n}}| + |e^{i\frac{\pi z}{n}}|}{|e^{-i\frac{\pi z}{n}}| - |e^{i\frac{\pi z}{n}}|} \times \frac{|e^{-i\pi z}| + |e^{i\pi z}|}{|-e^{-i\pi z}| - |e^{i\pi z}|}\right| \\
&= \frac{1}{N^2}\left|\coth\frac{\pi y}{m}\coth\frac{\pi y}{n}\coth\pi y\right| \qquad (y = iz) \\
&= \frac{1}{N^2}\coth|\frac{\pi y}{m}|\coth|\frac{\pi y}{n}|\coth|\pi y| \\
&\leq \frac{1}{N^2}\coth\frac{\pi}{2m}\coth\frac{\pi}{2n}\coth\frac{\pi}{2} \qquad (\because |z| > \frac{1}{2})
\end{aligned}$$

On vertical sides, since we have $z = \pm(N + \frac{3}{4}) + it$, we can compute that $|f(z)|$ is bounded by $\frac{M}{N^2}$ by some constant M. Thus for any z on any C_N, there is a constant A such that $|f(z)| \leq \frac{A}{N^2}$. Since the magnitude of the integral is less than the maximum of the integrand multiplied by the length of the contour,

$$\begin{aligned}
\left|\oint_{C_N} f(z)dz\right| &\leq \frac{A}{N^2}4(2N + \frac{3}{2}) \\
&\to 0 \quad \text{as } N \to \infty
\end{aligned}$$

By the residue formula, the integral is also $2\pi i\times$ (the sum of residues of the enclosed poles). The possible poles are $k, n(k + \frac{1}{2})$, $m(k + \frac{1}{2})$ when $k \in \mathbf{Z}$. When $k = 0$, residue equals to

$$\lim_{z\to 0}\frac{1}{z}\tan(\frac{\pi z}{m})\tan(\frac{\pi z}{n})\cot(\pi z) = \lim_{z\to 0}\frac{\sin\frac{\pi z}{m}\sin\frac{\pi z}{n}\cos\pi z}{z\cos\frac{\pi z}{m}\cos\frac{\pi z}{n}\sin\pi z} = \frac{\pi}{mn}$$

When $k \neq 0$, residue equals to $\frac{1}{\pi k^2}\tan(\frac{\pi k}{m})\tan(\frac{\pi k}{n})$. Consider when $z_k = m(k + \frac{1}{2})$. Then

$$\lim_{z\to z_k}\frac{1}{z^2}\tan(\frac{\pi z}{m})\tan(\frac{\pi z}{n})\cot(\pi z) = \frac{1}{z_k^2}\lim_{z\to z_k}\frac{\sin\frac{\pi z}{m}\cos\pi z}{\cos\frac{\pi z}{m}\sin\pi z}\tan\frac{\pi z}{n} = \frac{m}{z_k^2}\lim_{z\to z_k}\tan\frac{\pi z}{n}$$

Thus, z_k is a pole of $f(z)$ if singularity of $\tan\frac{\pi z}{n}$, i.e. z_k is a form of $n(k' + \frac{1}{2})$. Then because $m(2k+1) = n(2k'+1)$, poles of this form are $z_k = \frac{mn}{g}(k + \frac{1}{2})$. ($g = gcd(m, n)$) And residues at z_k is

$$\begin{aligned}
\lim_{z\to z_k}\frac{z - z_k}{z^2}\tan(\frac{\pi z}{m})\tan(\frac{\pi z}{n})\cot(\pi z) &= \frac{1}{z_k^2}\lim_{z\to z_k}\frac{(z - z_k)\sin\frac{\pi z}{m}\sin\frac{\pi z}{n}\cos\pi z}{\cos\frac{\pi z}{m}\cos\frac{\pi z}{n}\sin\pi z} \\
&= \frac{1}{z_k^2}\lim_{z\to z_k}\frac{\sin\frac{\pi z}{m}\sin\frac{\pi z}{n}(-\pi\sin\pi z)}{(-\frac{\pi}{m})\sin\frac{\pi z}{m}(-\frac{\pi}{n})\sin\frac{\pi z}{n}\sin\pi z} \\
&= -\frac{mn}{\pi z_k^2} = -\frac{4g^2}{\pi mn}\frac{1}{(2k+1)^2}
\end{aligned}$$

where $g = gcd(m, n)$. So

$$0 = (\text{sum of residues}) = \frac{\pi}{mn} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{1}{\pi k^2}\tan(\frac{\pi k}{m})\tan(\frac{\pi k}{n}) - \sum_{k=-\infty}^{\infty}\frac{4g^2}{\pi mn}\frac{1}{(2k+1)^2}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k^2}\tan(\frac{\pi k}{m})\tan(\frac{\pi k}{n}) &= \frac{\pi}{2}\left(\sum_{k=-\infty}^{\infty}\frac{4g^2}{\pi mn}\frac{1}{(2k+1)^2}\right)-\frac{\pi^2}{2mn}\\ &= \frac{4g^2}{mn}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}-\frac{\pi^2}{2mn}\\ &= \frac{4g^2}{mn}\frac{\pi^2}{8}-\frac{\pi^2}{2mn}=\frac{\pi^2}{2mn}(g^2-1)\end{aligned}$$

where g is the greatest common divisor of m and n. $\square$