POW 2012-15 Functional Equation

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pf) We can easily obtain that $f(0) = 0, f(x) = -f(-x), f(x^{n+1}) = xf(x)^n$. Let's consider two cases.
(1) n is odd
$n = 2k-1 (k \geq 1)$. By substituting y as $-y$, we obtain the following two equations:
$f(x^{2k} - y^{2k}) = (x-y)(f(x)^{2k-1} + f(x)^{2k-2}f(y) + \cdots + f(y)^{2k-1})$
$f(x^{2k} - y^{2k}) = (x+y)(f(x)^{2k-1} - f(x)^{2k-2}f(y) + \cdots - f(y)^{2k-1})$

Since two equations are equal, we obtain following with some calculation:
$(xf(y) - yf(x))(f(x)^{2k-2} + f(x)^{2k-4}f(y)^2 + \cdots + f(y)^{2k-2}) = 0 \cdots (*)$

Using $f(x^{n+1}) = xf(x)^n$, $f(1) = f(1)^n$. If $n = 1$, $f(1)$ could be arbitrary real number. From (*), we get $f(x) = f(1)x$ by substituting $y = 1$. If $n \geq 3$, $f(1) = 0$ or 1 or -1. When $f(1) = 0$, we substitute $y = 1$ for (*) again. We get $f(x) \equiv 0$. The other cases, when $f(1) = 1$ or -1, we put $y = -1$ in (*). Since $f(x)^{2k-1} + f(x)^{2k-2}f(1) + \cdots + f(1)^{2k-1} > 0$, $f(x) = f(1)x$. As a result, the solution is $f(x) \equiv cx (\forall c \in \mathbb{R})(n = 1)$, $f(x) \equiv 0$ or $f(x) \equiv x$ or $f(x) \equiv -x (n \geq 3)$.

(2) n is even
$n = 2k (k \geq 0)$. We can write original equation as $f(x^{2k+1} - y^{2k+1}) = (x - y)(f(x)^{2k} + f(x)^{2k-1}f(y) + \cdots + f(y)^{2k})$. Since $f(x) = -f(-x)$, we only consider when $x > 0$. We want to show that $f(x) \equiv x, f(x) \equiv 0$ are the solution. First, we put $-x$ instead of y. Then, $f(2x^{2k+1}) = 2xf(x)^{n+1} = 2f(x^{2k+1})$. Since x^{2k+1} can express all real number, $f(2x) = 2f(x)$. And if $x > 0$, $f(x) = x^{\frac{1}{2n+1}} f(x^{\frac{1}{2n+1}})^{2k} > 0$. Similarly, if $x < 0$, then $f(x) < 0$. Due to $f(x^{2k+1}) = xf(x)^{2k}$, $f(1) = 0$ or $f(1) = 1$.

Case1. $f(1) = 0$

We put $x' = x^{\frac{1}{2k+1}}, y' = (x-1)^{\frac{1}{2k+1}}$ instead of x,y in the equation. So, we can know that $0 = (x' - y')(f(x')^{2k} + f(x')^{2k-1}f(y') + \cdots + f(y')^{2k})$. We use that $a^{2k} + a^{2k-1}b + \cdots + b^{2k} \geq 0$ and equality holds if and only if $a = b = 0$. So we obtain $f(x') = 0$. Thus, for all $x \in \mathbb{R}$, $f(x) \equiv 0$.

Case2. $f(1) = 1$

Suppose that there is a positive number x such that $f(x) < x$. And define the set $A = \{x : f(x) < x, x > 0\}$. We will prove that such x does not exist. If $x \in A$, $x^{2k+1} \in A$ because $f(x^{2k+1}) = xf(x)^{2k} < x^{2k+1}$. And if $x \in A$, $x^{\frac{1}{2k+1}} \in A$ because $f(x) = x^{\frac{1}{2k+1}} f(x^{\frac{1}{2k+1}})^{2k} < x$. Also by using $f(2x) = 2f(x)$, we can know that if $x \in A$, $2x, \frac{x}{2} \in A$.When $x, y(x > y) \in A$, $f(x^{2k+1} - y^{2k+1}) = (x - y)(f(x)^{2k} + f(x)^{2k-1} f(y) + \cdots + f(y)^{2k}) < (x - y)(x^{2k} + x^{2k-1} y + \cdots + y^{2k}) = x^{2k+1} - y^{2k+1}$.

(i) $x > 1$

Put $y = 1$ in the original equation. We can obtain $\frac{f(x^{2k+1}-1)}{x^{2k+1}-1} = \frac{f(x)^{2k}+f(x)^{2k-1}+\cdots+1}{x^{2k}+x^{2k-1}+\cdots+1} < 1$. Therefore, $x^{2k+1} - 1 \in A$.

$x \in A \Rightarrow x^{\frac{1}{2k+1}} \in A \Rightarrow x - 1 \in A \Rightarrow (x - 1)^{\frac{1}{2k+1}} \in A$. We put $x^{\frac{1}{2k+1}}, (x - 1)^{\frac{1}{2k+1}}$ in the inequality $f(x^{2k+1} - y^{2k+1}) < x^{2k+1} - y^{2k+1}$. So, we gain $f(1) < 1$. Contradiction.

(ii) $x < 1$

Similarly, we obtain $\frac{f(1-x^{2k+1})}{1-x^{2k+1}} = \frac{f(x)^{2k}+f(x)^{2k-1}+\cdots+1}{x^{2k}+x^{2k-1}+\cdots+1} < 1$. Therefore, if $x \in A$, $1 - x^{2k+1} \in A$. $y \in A \Rightarrow y^{\frac{1}{2k+1}} \in A \Rightarrow 1 - y, (1 - y)^{\frac{1}{2k+1}} \in A$ Now, if $y \in A$, then $\frac{y}{2} \in A$. By some operations, we obtain $(\frac{y}{2})^{\frac{1}{2k+1}} \in A, 1 - \frac{y}{2}, 2 - y, (2 - y)^{\frac{1}{2k+1}} \in A$. We put $(2 - y)^{\frac{1}{2k+1}}$ and $(1 - y)^{\frac{1}{2k+1}}$ in the inequality $f(x^{2k+1} - y^{2k+1}) < x^{2k+1} - y^{2k+1}$. Then, we get $f(1) < 1$. Contradiction.

By (i),(ii), there is no $x > 0$ such that $f(x) < x$. Similarly we can prove that there is no $x > 0$ such that $f(x) > x$. Therefore, in this case, $f(x) \equiv x$.

In these cases, we can simply write $f(x) = f(1)x(f(1) = 0 or 1)$.

Answer:
$f(x) \equiv f(1)x (n = 1)$
$f(x) \equiv x$ or $f(x) \equiv -x$, $f(x) \equiv 0$ $(n \geq 3, n$:odd)
$f(x) \equiv f(1)x(f(1) = 0 or 1)(n$:even)