POW 2012 - 14

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Define $F(t) = \int_1^t f(x)dx$. Then, by the Fundamental Theorem of Calculus, we have F is differentiable and it's derivative is equal to f. i.e. $F'(t) = f(t)$. From the given equation, we have $F(t^3) = 3F(t)$. When we differentiate both side of that equation, we have $t^2 f(t^3) = f(t)$ or $f(t^3) = f(t)/t^2$.
Now, fix $t \in (0, \infty)$. Then we have $f(t) = \frac{f(t^{\frac{1}{3^n}})}{t^{2/3+2/3^2+\cdots+2/3^n}}$ for all $n \in \mathbb{N}$. Take a limit in both sides by n, then we have $f(t) = f(1)/t$. Actually, when we calculate original equation with $f(t) = f(1)/t$, we can verify the equation holds.