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(Dividing a circle) Let f be a continuous function from $[0,1]$ to a circle. Prove that there exists two closed intervals $I_1, I_2 \subseteq [0,1]$ such that $I_1 \cap I_2$ has at most one point, $f(I_1)$ and $f(I_2)$ are semicircles, and $f(I_1) \cup f(I_2)$ is a circle.

Proof. Let the circle C. I will directly construct such intervals. Note that continuous function f preserves connectedness and compactness. Since $[0,1]$ and C is bounded, every closed subspace of them is compact.

Step1. We find minimal interval whose image is the circle.

We consider $A = \{x \in [0,1] : f([0,x]) = C\}$. It is nonempty since $1 \in A$. We will show that $d = \inf A \in A$. Assume the contrary that $f([0,d]) \neq C$. Let $R \in C - f([0,d])$ By definition of A, $R \in f([d, d+(1-d)/n])$ for all $n \in \mathbb{N}$. Let $x_n \in [d, d+(1-d)/n]$ and $f(x_n) = R$. Then by continuity of f, $f(d) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = R$. This contradicts to $R \in C - f([0,d])$. By same argument on $B = \{x \in [x,d] : f([x,d]) = C\}$ with $0 \in B$, we can take $a = \sup B \in B$. Let $f(a) = Q$ and P be the opposite point of Q.

Claim. $f(a), f(d) \notin f((a,d))$, $f(a) = f(d) = Q$.

Assume the contrary that $f(a) \in f((a,d)), \exists r \in (a,d)$ such that $f(r) = f(a)$. We will use the fact that any $x > a$ cannot satisfy $f([x,d]) = C$ to verify this assumption is impossible.
$f([a,r]) \neq C$. Since $f([a,r])$ is closed, it contains boundary points. If $f([a,r])$ is singleton, $f([r,d]) = C$. So $f([a,r])$ is not singleton, so contains 2 boundary points. Let $b_1 < b_2 \in [a,r]$ such that $\{f(b_1), f(b_2)\} = \operatorname{bdy} f([a,r])$. Then $f([a,r]) = f([b_1,b_2])$. So $f([b_1,d]) = C$. That is, $a = b_1$. So $f(a)(= f(r))$ and $f(b_2)$ are distinct boundary points, we deduce that $f([a,b_2]) = f([b_2,r])$. So $f([b_2,d]) = C$. That is, $a = b_2$. Then $f([a,r]) = f([b_1,b_2])$ is singleton, and this is contradiction. That is, there is no such r. Finally, $Q \notin f((a,d))$. By similar argument with minimality of d in A, $f(d) \notin f((a,d))$.

Since (a,d) is connected, $f((a,d))$ is connected so $C - f((a,d))$ is at most singleton. Thus, we have $f(a) = f(d) = Q$ and $Q \notin f((a,d))$.

Step2. We cut the path to get two semicircles.

Let $D = f^{-1}(P) \cap [a,d]$. This is nonempty because $f([a,d]) = C$. Since f is continuous and $C - \{P\}$ is open, $f^{-1}(C - \{P\})$ is open. That is, $D = [a,d] - f^{-1}(C - \{P\})$ is closed. So $b = \inf D, c = \sup D \in D$. Then $S_1 = f([a,b]), S_2 = f([c,d])$ are semicircles containing P and Q because $f((a,b)), f((c,d))$ do not contain P and Q. $[a,b] \cap [c,d]$ has at most one point because $b = \inf D \leq$

$\sup D = c$. If $S_1 = S_2, f([a,c]) = f([a,d]) = C$. This contradicts to minimality of d. So $S_1 \neq S_2$, which means $S_1 \cup S_2 = C$.