Big partial sum

Minjae Park

POW2012-12. *Let A be a finite set of complex numbers. Prove that there exists a subset B of A such that*

$$\left|\sum_{z\in B} z\right| \geq \frac{1}{\pi}\sum_{z\in A}|z|.$$

Solution. For a complex number z, denote $\vec{z} = (\operatorname{Re} z, \operatorname{Im} z)$ be a vector in the complex plane. Let $B(\theta) = \{z \mid \vec{z}\cdot\omega(\theta) > 0\}$ where $\omega(\theta) = (\cos\theta, \sin\theta)$. Then,

$$\begin{aligned}\int_0^{2\pi}\sum_{z\in B(\theta)}\vec{z}\cdot\omega(\theta)d\theta &= \sum_{z\in A}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\vec{z}\cdot\omega(\arg\vec{z}+\phi)d\phi \\ &= \sum_{z\in A}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\|\vec{z}\|\cos\phi d\phi = 2\sum_{z\in A}|z|\end{aligned}$$

Therefore, by the mean value theorem, there exists Θ such that

$$\sum_{z\in B(\Theta)}\vec{z}\cdot\omega(\Theta) \geq \frac{1}{\pi}\sum_{z\in A}|z|.$$

Let $B = B(\Theta)$. Then, $\left|\sum_{z\in B} z\right| \geq \omega(\Theta)\cdot\sum_{z\in B}\vec{z} \geq \frac{1}{\pi}\sum_{z\in A}|z|$. The constant $\frac{1}{\pi}$ is maximal. Consider $A = \{z \mid z = e^{\frac{k\pi i}{2^{n-1}}}, k = 0, 1, \cdots, 2^n - 1\}$ as $n \to \infty$. □