POW2012-10 Platonic Solids

Minjae Park

POW2012-10. *Determine all Platonic solids that can be drawn with the property that all of its vertices are rational points.*

Solution. It is trivial that the cube can be drawn in the lattice of rational numbers, for instance with the vertices $(0,0,0),(0,0,1),\cdots,(1,1,0),(1,1,1)$. By taking the four points $(0,0,0),(0,1,1),(1,0,1),(1,1,0)$ among them, a tetrahedron also can be drawn. If we take the dual of the cube to make an octahedron which vertex is the center of mass of each face of the cube, then the octahedron also can be drawn so that all its vertices are rational points.

Now, let's prove that the dodecahedron and the icosahedron are impoossible to be drawn to satisfy the given property. Since an icosahedron is the dual of a dodecahedron, it is enough to show that every dodecahedron is not embeddable in the lattice of rational numbers. Assume that such an embedding is possible so that every vertex of a dodecahedron is rational point. Let's consider one of its pentagonal face and label the vertices to be $A, B, \cdots, E$ in a clockwise order. Then, by the assumtion, $AB^2, AC^2 \in \mathbf{Q}$. On the other hand, it is well known that $AC = \phi AB$ where $\phi = \frac{1+\sqrt{5}}{2}$ is a golden ratio. Thus, AB^2 and AC^2 cannot be both rational because ϕ^2 is not rational, which is a contradiction.

Therefore, the answer of the problem is the tetrahedron, the cube, and the octahedron. $\square$

Remark. In fact, the answer may be more specified. I will just state the propositions because I don't have enough time. (I will add the proof later.)

First, it doesn't matter although the restriction of rational points is altered to integral points because there is one-to-one correspondence with the proper translation and stretching. Thus, let's consider Platonic solids in the (integral) lattice.

Note that every cube contains a tetrahedron inside itself, by the natural way that I showed in the first example. Because a cube can be constructed with the same way from a tetrahedron, there is one-to-one correspondence between every cube and every tetrahedron in the lattice. In addition, we already know that there is a duality between the cube and the octahedron (we may need a dialation for the case of lattice), hence it is enough to consider only with the tetrahedron in the lattice. There is the following result about the tetrahedron in the lattice without a proof.

Theorem. *A tetrahedron is embeddable in the lattice if and only if the length of its side is in* $\mathbf{Z}\sqrt{2}$.

Therefore, the revisited answer of the original problem is: **a tetrahedron with side length $\mathbf{Q}\sqrt{2}$, a cube with side length $\mathbf{Q}$, and a octahedron with side length $\mathbf{Q}\sqrt{2}$.**