POW 2012-7 Product of Sine

proof.

Claim $n \in \mathbb{N}$.

$$\prod_{k=1}^{n} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}$$

proof of **Claim**. By $f(x) = \frac{x^n-1}{x-1} = \sum x^k = \prod(1 - e^{\frac{2k\pi}{n}})$ and half angle formulae, $f(1) = n = \prod(1 - e^{\frac{2k\pi}{n}}) = 2^{n-1}\prod(\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n})\sin\frac{k\pi}{n}$. Since $|\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n}| = 1$ and $\sin\frac{k\pi}{n} > 0$, $\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}$.

Let's solve original problem. Define $x_n = \prod \sin\frac{k\pi}{2n}$. Since $\sin\frac{k\pi}{2n} = \cos\frac{n-k\pi}{2n}$, $x_n = \prod \sin\frac{k\pi}{2n} = \prod \cos\frac{k\pi}{2n}$. Using this fact and double angle formulae, ${x_n}^2 = \prod \sin\frac{k\pi}{2n}\cos\frac{k\pi}{2n} = \frac{1}{2^{n-1}}\prod_{k=1}^{n}\sin\frac{k\pi}{n}$. By **Claim**, we obtain ${x_n}^2 = \frac{n}{4^{n-1}}$. Thus, $x_n = \frac{\sqrt{n}}{2^{n-1}}$.

We want to know some c such that $\lim_{n\to\infty}\frac{x_n}{c^n} = \lim_{n\to\infty}\frac{2\sqrt{n}}{(2c)^n}$ exist. Obviously, $2c \neq 1$. If $0 < 2c < 1$, $\frac{2\sqrt{n}}{(2c)^n} > \sqrt{n}$. Then, the limit does not exist. When $2c > 1$, by binomial theorem, $\frac{\sqrt{n}}{(1+\epsilon)^n} \leq \frac{\sqrt{n}}{1+n\epsilon} \leq \frac{1}{\sqrt{n}\epsilon}(1+\epsilon = 2c)$. By squeeze theorem, its limit exists and the value of limit is zero. Thus, the set X is $\left\{c | c > \frac{1}{2}, c \in \mathbb{R}\right\}$. And the infimum of this set is $c = \frac{1}{2}$.