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(Rank of a matrix) Let M be an $n \times n$ matrix over the reals. Prove that $\operatorname{rank} M = \operatorname{rank} M^2$ if and only if $\lim_{\lambda \to 0} (\lambda I_n + M)^{-1} M$ exists.

Proof. Suppose that $\operatorname{rank} M = k$. Then $\exists \{B_1, \dots, B_k\} \subset \mathbb{R}^n$ a basis of the $\operatorname{row} M$. Let $B \in \mathbb{R}^{k \times n}$ with $B_1, \dots, B_k$ as rows. Then $M = AB$ for some $A \in \mathbb{R}^{n \times k}$ since $\operatorname{row} M = \operatorname{row} B$. Then $\operatorname{col} A = \operatorname{col} M$. So we get $\operatorname{rank} A = \operatorname{rank} B = k$.

Remark that $\exists \lim_{\lambda \to 0} X \Rightarrow \exists \lim_{\lambda \to 0} YX \wedge \exists \lim_{\lambda \to 0} XY$ for all constant matrix Y since every entry of XY and YX is a linear combination of some entries of X.

Step1. $\operatorname{rank} M^2 = \operatorname{rank} M \Leftrightarrow BA$ is invertible.

Note that $M^2 = (AB)(AB) = A(BA)B$. BA is invertible $\Leftrightarrow \operatorname{ran}(BA)B = \operatorname{ran} B$, then $\operatorname{ran} M^2 = \operatorname{ran} AB = \operatorname{ran} M$.

Step2. $\exists \lim_{\lambda \to 0} (\lambda I_n + M)^{-1} M \Leftrightarrow \exists \lim_{\lambda \to 0} (\lambda I_k + BA)^{-1}$

Now we use the formula

$$(\lambda I_n + AB)^{-1} = \lambda^{-1}[I_n - A(\lambda I_k + BA)^{-1} B]$$

This is easily verified by matrix multiplication. So

$$(\lambda I_n + M)^{-1} M = (\lambda I_n + AB)^{-1} AB = I_n - \lambda(\lambda I_n + AB)^{-1} = A(\lambda I_k + BA)^{-1} B$$

Since $\operatorname{rank} A = \operatorname{rank} B = k$, $\operatorname{col} B = \mathbb{R}^n$. That is, $\exists B' \in \mathbb{R}^{n \times k}$ such that $BB' = I_k$. By similar argument on $\operatorname{row} A$, $\exists A' \in \mathbb{R}^{k \times n}$ such that $A'A = I_k$. So $(\lambda I_k + BA)^{-1} = A'A(\lambda I_k + BA)^{-1} BB'$

Therefore, $\exists \lim_{\lambda \to 0} A(\lambda I_k + BA)^{-1} B \Leftrightarrow \exists \lim_{\lambda \to 0} A'A(\lambda I_k + BA)^{-1} BB' = \lim_{\lambda \to 0} (\lambda I_k + BA)^{-1}$

Step3. $\exists \lim_{\lambda \to 0} (\lambda I_k + BA)^{-1} \Leftrightarrow BA$ is invertible.

Note that $(\lambda I_k + BA)^{-1} = \frac{1}{\det(\lambda I_k + BA)} \operatorname{adj}(\lambda I_k + BA)$. $\lim_{\lambda \to 0} \operatorname{adj}(\lambda I_k + BA) = \operatorname{adj} BA$ and $\lim_{\lambda \to 0} \det(\lambda I_k + BA) = \det BA$ because determinant and entries of the adjoint matrix are composed of product and sum of some entries from original matrix. So, $\exists \lim_{\lambda \to 0} (\lambda I_k + BA)^{-1} \Leftrightarrow \det BA \neq 0 \Leftrightarrow BA$ is invertible.

Consequently, $\operatorname{rank} M = \operatorname{rank} M^2 \Leftrightarrow \exists \lim_{\lambda \to 0} (\lambda I_n + M)^{-1} M$