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(Platonic solids) Determine all platonic solids that can be drawn with the property that all of its vertices are rational points.

Proof. For first three platonic solids, it is easy to find specific examples with rational coordinates.

Tetrahedron : $(0,0,0)(0,1,1)(1,0,1)(1,1,0)$

Cube : $(0,0,0)(1,0,0)(0,1,0)(0,0,1)(1,1,0)(1,0,1)(0,1,1)(1,1,1)$

Octahedron : $(1,0,0)(-1,0,0)(0,1,0)(0,-1,0)(0,0,1)(0,0,-1)$

Claim. There is no regular pentagon with rational coordinates.

Assume the contrary that there exists a regular pentagon $ABCDE$ with such property. Let $\vec{X}$ be a coordinate vector of point X. Consider $\theta = \angle ABC$

$$\cos\theta = \frac{(\vec{A}-\vec{B})\cdot(\vec{C}-\vec{B})}{|\vec{A}-\vec{B}||\vec{C}-\vec{B}|}$$

So

$$\cos^2\theta = \frac{[(\vec{A}-\vec{B})\cdot(\vec{C}-\vec{B})]^2}{(\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})\times(\vec{C}-\vec{B})\cdot(\vec{C}-\vec{B})}$$

which is rational number because dot product of two vectors with rational coordinates is rational.

But $\cos\theta = \cos\frac{2\pi}{5} = \frac{1}{4}(\sqrt{5}-1)$, so $\cos^2\theta = \frac{1}{8}(3-2\sqrt{5})$ is not rational.

Therefore there is no dodecahedron or icosahedron with rational coordinates since they contain regular pentagons composed of their vertices.

Consequently, tetrahedron, cube, octahedron are only platonic solids can be drawn with rational coordinates.