

Protected: POW 2012-5 Iterative geometric mean

For given positive real numbers a_1, \dots, a_k and for each integer $n \geq k$, let a_{n+1} be the geometric mean of $a_n, a_{n-1}, a_{n-2}, \dots, a_{n-k+1}$. Prove that $\lim_{n \rightarrow \infty} a_n$ exists and compute this limit.

Solution. Let $x_n \equiv \ln a_n$. Then, x_{n+1} is the arithmetic mean of $x_n, x_{n-1}, \dots, x_{n-k+1}$. Each x_n is a linear combination of x_1, x_2, \dots, x_k , so let $x_n = \sum_{j=1}^k p_{nj} x_j$.

Next, define $M_n = [m_{ij}]_{k \times k}$ be a square matrix such that $m_{ij} = p_{(n+i-1)j}$ for each $1 \leq i, j \leq k$. Note that $M_1 = I_k$,

$$M_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1/k & 1/k & 1/k & \cdots & 1/k \end{pmatrix}_{k \times k},$$

and $M_{n+1} = M_2 M_n = M_2^n$. Therefore, the given problem is exactly same with finding $\lim_{n \rightarrow \infty} M_2^n$.

It is remarkable that M_2 is a (right) stochastic matrix, so $M_2 \mathbf{1} = \mathbf{1}$ when $\mathbf{1}$ is a column vector which every element is 1, thus 1 is an eigenvalue of M_2 . Also, if r is the Perron root of M_2 , then by the inequality

$$1 = \min_i \sum_j m_{ij} \leq r \leq \max_i \sum_j m_{ij} = 1$$

$r = 1$ if it exists. Since $M_{k+1} = M_2^{k+1}$ is a positive matrix, the Perron-Frobenius theorem implies that r and $\lim_{n \rightarrow \infty} M_2^n$ exist.

If there is the unique row vector π satisfying $\pi M_2 = \pi$, then $\lim_{n \rightarrow \infty} [M_2^n]_{ij} = \pi_j$. Such row vector should satisfy $\pi(M_2 - I_k) = 0$. As 1 is an eigenvalue, $\det(M_2 - I_k) = 0$, but if we consider the principal minor determinant, $\text{rank}(M_2 - I_k) = k - 1$. Consequently, because a row vector $(1, 2, \dots, k)$ is a solution of $\pi(M_2 - I_k) = 0$ and $\text{nulity}(M_2 - I_k) = 1$, every candidate of π is a form of $c(1, 2, \dots, k)$ for some constant c . On the other hand, since $M_n \mathbf{1} = M_2^n \mathbf{1} = \mathbf{1}$ for all n , there is only one possibility restricted by $\sum_j \pi_j = 1$, so $c = \frac{2}{k(k+1)}$. Hence, $\lim_{n \rightarrow \infty} p_{nj} = \lim_{n \rightarrow \infty} [M_2^n]_{ij} = \pi_j = \frac{2j}{k(k+1)}$.

In conclusion, $\lim_{n \rightarrow \infty} x_n = \sum_{j=1}^k \frac{2j x_j}{k(k+1)}$, thus

$$\lim_{n \rightarrow \infty} a_n = \prod_{j=1}^k a_j^{\frac{2j}{k(k+1)}}.$$