KAIST POW 2012-3 Integral

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Compute

$$f(x) = \int_0^1 \frac{\log(1-2t\cos x + t^2)}{t}\,dt.$$

Solution.

We need a simple manipulation first to the original function,

$$1-2t\cos x + t^2 = \cos^2 x + \sin^2 x - 2t\cos x + t^2 = \sin^2 x + (1-\cos x)^2$$

Then, differentiate the function f(x) and we gain

$$f'(x) = 2\sin x \int_0^1 \frac{1}{(t-\cos x)^2 + \sin^2 x}\,dt = 2\int_0^1 \frac{\frac{1}{\sin x}}{1+(\frac{t-\cos x}{\sin x})^2}\,dt = \left[\tan^{-1}(\frac{t-\cos x}{\sin x})\right]_0^1$$

Therefore,

$$f'(x) = \tan^{-1}(\frac{1-\cos x}{\sin x}) - \tan^{-1}(-\frac{\cos x}{\sin x})$$

Because $\tan x = \dfrac{2\tan\frac{x}{2}}{1-(\tan\frac{x}{2})^2}$ *, there's a quadratic equation that* $\tan\frac{x}{2}$ *is a variable,*

$$\tan x(\tan\frac{x}{2})^2 + 2\tan\frac{x}{2} - \tan x = 0$$

$$\therefore\ \tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$$

Also,

$$-\frac{\cos x}{\sin x} = \frac{\cos x}{\sin x} = \cot x = \tan(\frac{\pi}{2} - x)$$

After the Substitute of those two formula,

$$f'(x) = 2(\frac{x}{2}) + 2(\frac{\pi}{2} - x) = \pi - x$$

$$\therefore\ f(x) = \pi x - \frac{x^2}{2} + C$$

Now, we have to find the constant C for completing the exact f(x).

$$f(\frac{x}{2}) + f(\pi - \frac{x}{2}) = \int_0^1 \frac{\log(1-2t\cos\frac{x}{2}+t^2)}{t}\,dt + \int_0^1 \frac{\log(1+2t\cos\frac{x}{2}+t^2)}{t}\,dt$$

$$= \int_0^1 \frac{\log(1-2t^2\cos x + t^4)}{t}\,dt = \frac{1}{2}f(x)$$

(Integration by substitution $u=t^2$*)*

When we put $x=\pi$, $2f(\frac{\pi}{2}) = \frac{1}{2}f(\pi)$ *i.e.* $2(\frac{\pi^2}{2} - \frac{\pi^2}{8} + C) = \frac{1}{2}(\pi^2 - \frac{\pi^2}{2} + C)$ $\therefore\ C = -\frac{\pi^2}{3}$

$$\therefore\ f(x) = \pi x - \frac{x^2}{2} - \frac{\pi^2}{3}$$