Protected: POW 2012-3 Integral

Let $\phi(x) = \int_0^1 \frac{\ln(1-2t\cos x + t^2)}{t}dt$ be a function of $x \in [0, 2\pi)$. Then, by the Leibniz integration rule,

$$\phi'(x) = \frac{d}{dx}\int_0^1 \frac{\ln(1-2t\cos x+t^2)}{t}dt = \int_0^1 \frac{\partial}{\partial x}\frac{\ln(1-2t\cos x+t^2)}{t}dt$$

$$= \int_0^1 \frac{2\sin x}{(t-\cos x)^2+\sin^2 x}dt = 2\left(\tan^{-1}\frac{1-\cos x}{\sin x} + \tan^{-1}\frac{\cos x}{\sin x}\right)$$

$$= 2\tan^{-1}\frac{\sin x}{1-\cos x} = 2\tan^{-1}\cot\frac{x}{2} = \pi - x.$$

Therefore, $\phi(x) = \pi x - \frac{1}{2}x^2 + C$ for some constant C. To evaluate this constant, let's consider the function value at $x = \pi$. First note that $\ln(1+t) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}t^n}{n}$ for $0 \le t \le 1$, thus

$$\int_0^1 \frac{\ln(1+t)}{t}dt = \sum_{n=1}^{\infty}\int_0^1 \frac{(-1)^{n+1}t^{n-1}}{n}dt = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - 2\sum_{n=1}^{\infty}\frac{1}{(2n)^2}.$$

The last equality holds by the absolute convergence, and it is well known that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. As a result,

$$\phi(\pi) = \frac{\pi^2}{2} + C = \int_0^1 \frac{2\ln(1+t)}{t}dt = \frac{\pi^2}{6},$$

so we conclude that $\phi(x) = \pi x - \frac{1}{2}x^2 - \frac{\pi^2}{3}$. Since $\phi(x)$ can be extended to every real numbers as a periodic function,

$$f(x) = \int_0^1 \frac{\ln(1-2t\cos x+t^2)}{t}dt = \pi(x \mod 2\pi) - \frac{1}{2}(x \mod 2\pi)^2 - \frac{\pi^2}{3}.$$