

$$\begin{aligned}
 f(x) &= \int_0^1 \frac{\log(1-2tx\cos x + t^2)}{t} dt = \int_0^1 \frac{\log(1-(e^{ix}+e^{-ix})t+t^2)}{t} dt = \int_0^1 \frac{\log((1-e^{ix}t)(1-e^{-ix}t))}{t} dt \\
 &= \int_0^1 \frac{\log(1-e^{ix}t)}{t} dt + \int_0^1 \frac{\log(1-e^{-ix}t)}{t} dt \\
 &= \int_0^x \frac{e^{ix}}{t} \log(1-t) dt + \int_0^{-x} \frac{e^{-ix}}{t} \log(1-t) dt
 \end{aligned}$$

$$\begin{aligned}
 \text{Since } \log(1-t) &= \sum_{k=1}^{\infty} -\frac{1}{k} t^k, \quad \int_0^x \frac{e^{ix}}{t} \log(1-t) dt + \int_0^{-x} \frac{e^{-ix}}{t} \log(1-t) dt \\
 &= \left(\sum_{k=1}^{\infty} \frac{e^{ikx}}{k} \right) - \left(\sum_{k=1}^{\infty} \frac{e^{-ikx}}{k} \right) \quad \dots \textcircled{1}
 \end{aligned}$$

$|e^{ix}| < 1$, so $\textcircled{1}$ is absolute convergent

$$\therefore \textcircled{1} = - \sum_{k=1}^{\infty} \frac{e^{ikx} + e^{-ikx}}{k^2} = - \sum_{k=1}^{\infty} \frac{2 \cos(kx)}{k^2} = -2 \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} \quad \dots \textcircled{2}$$

Now, think about Bernoulli polynomials

$$B_1(x) = x - \frac{1}{2}$$

Let's calculate it in interval $[-\frac{1}{2}, \frac{1}{2}]$

For $f(x)$ with period T , Fourier expansion is

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{T}\right)$$

$$\text{where } a_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos\left(\frac{2\pi n x}{T}\right) dx \quad \text{and} \quad b_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \sin\left(\frac{2\pi n x}{T}\right) dx$$

$$\text{In } B_1(x), f(x) = x, a_0 = \frac{1}{2}$$

$$\text{Then } a_n = 0, b_n = -4 \cdot \int_0^{\frac{1}{2}} x \cdot \sin(2\pi n x) dx = -\frac{(-1)^{n+1}}{n\pi}$$

$$\text{For } [-\frac{1}{2}, \frac{1}{2}], x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(2\pi n(x - \frac{1}{2})) + \frac{1}{2}$$

$$B_1(x) = x - \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(2\pi n(x - \frac{1}{2})) \quad \text{in } [0, 1]$$

$$\text{Then, } B_2(x) - B_2(0) = \int_0^x B_1(t) dt = 2 \cdot \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^2} (\cos(2\pi n(x - \frac{1}{2})) - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^2} \cos(-\pi n)) \right)$$

$$B_2(x) = 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^2} \cdot (\cos(2\pi n(x - \frac{1}{2})))$$

$$= 2 \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{(2\pi n)^2} = \frac{1}{2\pi^2} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^2} \quad \text{in } [0, 1]$$

By comparing $\textcircled{2}$ and $\textcircled{3}$, we can get

$$f(x) = -2\pi^2 \cdot B_2\left(\frac{x}{2\pi}\right) = \frac{x^2}{2} - 2\pi x + \frac{\pi^2}{3} \quad \text{when } x \in [0, 2\pi]$$

So, $f(x)$ is periodic function with period 2π

$$\text{When } x \in [0, 2\pi], f(x) = \frac{x^2}{2} - 2\pi x + \frac{\pi^2}{3}$$

$$\begin{aligned}
 &= -2\pi^2 \left(\left(\frac{x}{2\pi}\right)^2 - \frac{x}{2\pi} + \frac{1}{6} \right) \\
 &= -\frac{x^2}{2} + \pi x - \frac{\pi^2}{3}
 \end{aligned}$$

