Problem of the Week 2012-3

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Problem: Compute

$$f(x) = \int_0^1 \frac{\log(1 - 2t\cos x + t^2)}{t} dt\,.$$

Solution:

$$\begin{aligned} f'(x) &= 2\sin x \int_0^1 \frac{1}{1 - 2t\cos x + t^2} dt \\ &= -i \int_0^1 \left[\frac{1}{t - e^{ix}} - \frac{1}{t - e^{-ix}} \right] dt \\ &= -i \left[\log\left(\frac{t - e^{ix}}{t - e^{-ix}} \right) \right]_{t=0}^{1} \\ &= -i \left[\log(-e^{ix}) - \log(e^{2ix}) \right] \\ &= -i \log[e^{i(\pi - x)}] \\ &= \pi - x\,. \qquad (0 < \mathrm{Re}(x) < 2\pi) \end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned} f(0) &= 2 \int_0^1 \frac{\log(1-t)}{t} dt \\ &= -2 \int_0^1 \sum_{n=1}^{\infty} \frac{t^{n-1}}{n} dt \\ &= -2 \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= -\frac{\pi^2}{3} \end{aligned} \tag{2}$$

Therefore,

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(y) dy \\ &= -\frac{\pi^2}{3} + \pi x - \frac{x^2}{2} \\ &= \frac{\pi^2}{6} - \frac{1}{2}(x - \pi)^2\,. \qquad (0 < \mathrm{Re}(x) < 2\pi) \end{aligned} \tag{3}$$

$f(x)$ is periodic in $\mathrm{Re}(x)$ with period 2π. This condition along with the above result determines the value of $f(x)$ in other regions of the complex plane.