Proof. Define $b_n := loga_n$. Then b_{n+1} is the arithmetic mean of $b_n, \cdots, b_{n-k+1}$. We want to check whether limit of b_n exists. So, it is enough to show that b_n is a Cauchy sequence.

Assume that $b_1, \cdots, b_k$ is contained for some interval of length x. Then it is obvious that b_i is contained in that interval for all i. Also, for all $i \geq k+1$,

$$b_{i+1} - b_i = \frac{b_i + \cdots + b_{i-k+1}}{k} - \frac{b_{i-1} + \cdots + b_{i-k}}{k} = \frac{b_i - b_{i-k}}{k}$$

Then for all i, j between $k+1$ and $2k$,

$$|b_i - b_j| = \left| \sum_{l=i}^{j-1} \frac{b_l - b_{l-k}}{k} \right| \leq \sum_{l=i}^{j-1} \left| \frac{b_l - b_{l-k}}{k} \right| \leq \sum_{l=i}^{j-1} \frac{x}{k} \leq \frac{k-1}{k} x$$

Therefore, $b_{k+1}, b_{k+2}, \cdots, b_{2k}$ is contained in some interval of length $\frac{k-1}{k}x$, and so is a_i for all $i \geq 2k+1$. In the same way, we can show that for any n, there is an interval of length $(\frac{k-1}{k})^n x$ which contains b_i's for all $i > nk$. Thus b_i is a Cauchy sequence and its limit exists. Then limit of $a_n = e^{b_n}$ exists.

From definition, we can observe that

$$\begin{aligned} b_{n+1} + 2b_{n+2} + \cdots + kb_{n+k} &= b_{n+1} + 2b_{n+2} + \cdots + (k-1)b_{n+k-1} + (b_n + \cdots + b_{n+k-1}) \\ &= b_n + 2b_{n+1} + \cdots + kb_{n+k-1} \\ &= \cdots\cdots \\ &= b_1 + 2b_2 + \cdots + kb_k \end{aligned}$$

Therefore, by taking the limit for each side,

$$b = \lim_{n\to\infty} b_n = \frac{2}{k(k+1)}(b_1 + \cdots + kb_k)$$

So we obtain

$$a = \lim_{n\to\infty} a_n = e^{\lim_{n\to\infty} b_n} = e^{\frac{2}{k(k+1)}(b_1 + \cdots + kb_k)} = \sqrt[\frac{k(k+1)}{2}]{a_1 a_2^2 \cdots a_k^k}$$

□