Sum of squares

Minjae Park

POW2012-4. *Find the smallest and the second smallest odd integers n satisfying the following property: $n = x_1^2 + y_1^2$ and $n^2 = x_2^2 + y_2^2$ for some positive integers x_1, y_1, x_2, y_2 such that $x_1 - y_1 = x_2 - y_2$.*

Solution. The answer is 5 and 261. ($5 = 2^2 + 1^2, 5^2 = 4^2 + 3^2$, and $261 = 15^2 + 6^2, 261^2 = 189^2 + 180^2$.)

To verify this, suppose that $5 < n < 261$ satisfies the given property. Assume that $x_i > y_i$, and let $x_1 - y_1 = x_2 - y_2 = k$, then k must be odd. Since $261 > n > k^2$, $k = 1, 3, \cdots, 15$. Note that $n^2 = k^2 + 2x_2y_2 = k^2 + (x_2+y_2)^2 - n^2$, thus $2n^2 = k^2 + p^2$ has an integral solution.

(a) If $k = 1$, then by solving a Pell's equation $p^2 - 2n^2 = -1$, we obtain $p + n\sqrt{2} = (1+\sqrt{2})^{2j-1}$ for $j = 1, 2, 3, \cdots$, and $n = 1, 5, 29, 169, 985, \cdots$. It is easy to check that 29 and 169 are not representable as $y_1^2 + (y_1+1)^2$, so $k \neq 1$.

Note that $x_2 \equiv y_2 \mod k$, so $n^2 \equiv 2x_2^2 \mod k$. Hence, if $\left(\frac{2}{k}\right) \neq 1$ then $x_2 \equiv y_2 \equiv n \equiv x_1 \equiv y_1 \equiv 0 \mod k$.

(b) If $3|k$, since $\left(\frac{2}{3}\right) = -1$, we obtain $9|n$ from $n = x_1^2 + y_1^2$. As $n^2 = x_2^2 + y_2^2$, $x_2 \equiv y_2 \equiv 0 \mod 9$, therefore k is divisible by 9 which means that $k = 9$. Since $n = x_1^2 + y_1^2 < 261 = 6^2 + 15^2$, $y_1 = 3$ and $x_1 = 12$, so $n = 153$. However, $2n^2 - k^2 = 2\cdot 153^2 - 9^2 = 9^2(2\cdot 17^2 - 1) = 9^2 \cdot 577 = p^2$ does not have an integral solution. As a consequence, $k \neq 3, 9, 15$.

(c) If $k = 5, 11, 13$, then $\left(\frac{2}{k}\right) = -1$. Since $n < 261 < 10^2 + 15^2$, $x_1 = 10$ and $y_1 = 5$ is the only possibility if $k = 5$. In this case, $n = 125$, thus $2n^2 - k^2 = 5^2(2\cdot 5^4 - 1) = 5^2 \cdot 1249 = p^2$ does not have an integral solution, so $k \neq 5$. Also, $k \neq 11, 13$ because $n < 261 < 11^2 + 22^2 < 13^2 + 26^2$.

(d) If $k = 7$, $n < 261 < 8^2 + 15^2$, so $y_1 = 1, 2, \cdots, 8$ and all possible values are $n = 65, 85, 109, 137, 169, 205, 245$. It is not so hard to compute $2n^2 - k^2 = 2n^2 - 49 = p^2$ is never a square, therefore $k \neq 7$.

From (a) to (d), such n does not exist. $\square$