Constant function

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POW2011-23. *Let $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a function such that for each point $a \in R^n$, $\lim_{x\to a} \frac{\|f(x)-f(a)\|}{\|x-a\|}$ exists. Prove that f is a constant function.*

Solution. There are useful notations and results from Lipschitz analysis.

Definition. *A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be* **Lipschitz** *if*

$$\|f(x) - f(y)\| \le L\|x - y\|$$

for every $x, y \in \Omega$ where L is a non-negative constant.

Definition. *The* **pointwise Lipschitz constant** *of a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is*

$$\mathrm{Lip} f(a) \equiv \limsup_{x\to a} \frac{\|f(x) - f(a)\|}{\|x - a\|}.$$

Theorem (Lebesgue). *Let $f : (a, b) \to \mathbb{R}$ be Lipschitz. Then f is differentiable at almost every point in (a, b).*

Theorem (Rademacher). *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. Then f is differentiable at almost every point in Ω.*

Theorem (Stepanov). *A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable almost everywhere in the set*

$$L(f) \equiv \{a \in \Omega : \mathrm{Lip} f(a) < \infty\}.$$

Let's go back to the original problem, let $h(a) \equiv \lim_{x\to a} \frac{\|f(x)-f(a)\|}{\|x-a\|}$. Then, $\mathrm{Lip} f(a) = h(a) < \infty$ for all $a \in R^n$, thus f is differentiable almost everywhere by Stepanov's theorem.

Suppose that f is differentiable at a. Then, the $(n-1) \times n$ Jacobian matrix $Df(a)$ exists. Since $\mathrm{rank}(Df(a)) \le n - 1$, $\mathrm{nullity}(Df(a)) \ge 1$, so there is a non-zero $n \times 1$ column vector $b \in \mathrm{Ker}(Df(a))$.

Let $p : \mathbb{R} \to \mathbb{R}^n$ be a smooth curve passing the point a, which is given by $p(t) = a + bt$. Then,

$$
\begin{aligned}
h(a)^2 &= \lim_{x \to a} \frac{\|f(x) - f(a)\|^2}{\|x - a\|^2} \\
&= \lim_{t \to 0} \frac{\|f(p(t)) - f(a)\|^2}{\|p(t) - a\|^2} \\
&= \lim_{t \to 0} \frac{\sum_{k=1}^{n-1} (f_k(p(t)) - f_k(a))^2}{\|b\|^2 t^2} \\
&= \lim_{t \to 0} \frac{\sum_{k=1}^{n-1} \left(\sum_{j=1}^{n} D_j f_k \frac{dp_j}{dt} \right) (f_k(p(t)) - f_k(a))}{\|b\| t} \\
&= \lim_{t \to 0} \frac{\sum_{k=1}^{n-1} Df(a)(b)_k (f_k(p(t)) - f_k(a))}{\|b\| t} \\
&= 0
\end{aligned}
$$

by L'Hospital's rule. Therefore, $h(a) = 0$ if f is differentiable at a.

Since f is differentiable almost everywhere, $h(a) = 0$ almost everywhere. If $h(a) = \lim_{x \to a} \frac{\|f(x) - f(a)\|}{\|x - a\|} \neq 0$ for some a, then the measure of set $D = \{a : h(a) \neq 0\}$ is not zero, which is a contradiction. As a result, $h(a) = 0$ for all $a \in \mathbb{R}^n$.

By the definition of derivative, $Df(a)$ exists and vanishes for all $a \in \mathbb{R}^n$ because if so,

$$
\lim_{x \to a} \frac{\|f(x) - f(a) - Df(a)(x - a)\|}{\|x - a\|} = h(a) = 0
$$

and such a linear map uniquely exists. In conclusion, by the mean value theorem, f is a constant function.

$\square$