Protected: POW 2012-2 Sum with a permutation

Let S_n^j be the set of all permutations on $\{1, 2, \cdots, n\} \setminus \{j\}$. I have a claim which can be proved by the induction.

Claim. With the same restriction (but, except for the condition that $\sum_{i=1}^n x_i$ is zero) of the original problem,

$$\sum_{\pi \in S_n} \prod_{k=1}^{n-1} \frac{1}{\sum_{i=1}^k x_{\pi(i)}} = \frac{\sum_{i=1}^n x_i}{\prod_{i=1}^n x_i}.$$

proof. Let's prove it by the induction. For $n = 2$, it is trivial. Suppose that the claim is true for some $n-1$ which is greater than 1. Next, to check that the claim is also true for n, see the following equalities.

$$\sum_{\pi \in S_n} \prod_{k=1}^{n-1} \frac{1}{\sum_{i=1}^k x_{\pi(i)}} = \sum_{j=1}^n \frac{1}{\sum_{i=1}^n x_i - x_j} \sum_{\pi \in S_n^j} \prod_{k=1}^{n-2} \frac{1}{\sum_{i=1}^k x_{\pi(i)}}$$

(by the induction hypothesis) $$= \sum_{j=1}^n \frac{1}{\sum_{i=1}^n x_i - x_j} \frac{\sum_{i=1}^n x_i - x_j}{\prod_{i=1}^n x_i / x_j} = \frac{\sum_{j=1}^n x_j}{\prod_{i=1}^n x_i}.$$

Therefore, the claim is verified by the induction. As a result, the only possible value of the original sum is 0.